*Stanislaw Lojasiewicz Lecture*

# Optimal Transportation in the Twenty First Century

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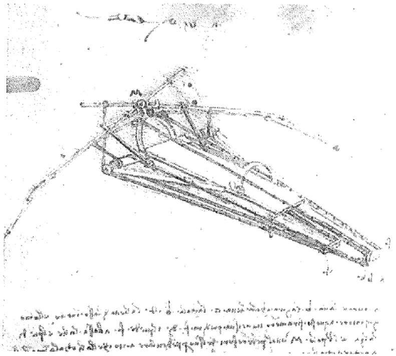
16 May, 2013

# Optimal transportation circa 1502

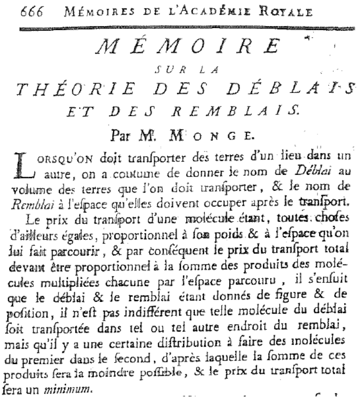


Leonardo da Vinci

*"The great bird will take flight above the ridge... filling the universe with awe, filling all writings with its fame..."*

# The Monge problem 1781

Gaspard Monge:



666 MÉMOIRES DE L'ACADÉMIE ROYALE

MÉMOIRE
SUR LA
THÉORIE DES DÉBLAIS
ET DES REMBLAIS.

Par M. MONGE.

LORSQU'ON doit tranſporter des terres d'un lieu dans un autre, on a coutume de donner le nom de *Déblai* au volume des terres que l'on doit tranſporter, & le nom de *Remblai* à l'eſpace qu'elles doivent occuper après le tranſport.

Le prix du tranſport d'une molécule étant, toutes choſes d'ailleurs égales, proportionnel à ſon poids & à l'eſpace qu'on lui fait parcourir, & par conſéquent le prix du tranſport total devant être proportionnel à la ſomme des produits des molécules multipliées chacune par l'eſpace parcouru, il s'enſuit que le déblai & le remblai étant donnés de figure & de poſition, il n'eſt pas indifférent que telle molécule du déblai ſoit tranſportée dans tel ou tel autre endroit du remblai, mais qu'il y a une certaine diſtribution à faire des molécules du premier dans le ſecond, d'après laquelle la ſomme de ces produits ſera la moindre poſſible, & le prix du tranſport total ſera un *minimum*.

# Kantorovich 1942

ON THE TRANSLOCATION OF MASSES

L. V. Kantorovich*

*The original paper was published in Dokl. Akad. Nauk SSSR,* **37**, No. 7–8, 227–229 (1942).

We assume that $R$ is a compact metric space, though some of the definitions and results given below can be formulated for more general spaces.

Let $\Phi(e)$ be a mass distribution, i.e., a set function such that: (1) it is defined for Borel sets, (2) it is nonnegative: $\Phi(e) \geq 0$, (3) it is absolutely additive: if $e = e_1 + e_2 + \cdots$; $e_i \cap e_k = 0$ $(i \neq k)$, then $\Phi(e) = \Phi(e_1) + \Phi(e_2) + \cdots$. Let $\Phi'(e')$ be another mass distribution such that $\Phi(R) = \Phi'(R)$. By definition, a translocation of masses is a function $\Psi(e, e')$ defined for pairs of $(B)$-sets $e, e' \in R$ such that: (1) it is nonnegative and absolutely additive with respect to each of its arguments, (2) $\Psi(e, R) = \Phi(e)$, $\Psi(R, e') = \Phi'(e')$.

Let $r(x, y)$ be a known continuous nonnegative function representing the work required to move a unit mass from $x$ to $y$.

We define the work required for the translocation of two given mass distributions as

$$W(\Psi, \Phi, \Phi') = \int_R \int_R r(x, x')\Psi(de, de') = \lim_{\lambda \to 0} \sum_{i,k} r(x_i, x'_k)\Psi(e_i, e'_k),$$

where $e_i$ are disjoint and $\sum_1^n e_i = R$, $e'_k$ are disjoint and $\sum_1^m e'_k = R$, $x_i \in e_i$, $x'_k \in e'_k$, and $\lambda$ is the largest of the numbers $\operatorname{diam} e_i$ $(i = 1, 2, \ldots, n)$ and $\operatorname{diam} e'_k$ $(k = 1, 2, \ldots, m)$.

Clearly, this integral does exist.

We call the quantity

$$W(\Phi, \Phi') = \inf_{\Psi} W(\Psi, \Phi, \Phi')$$

the minimal translocation work. Since the set of all functions $\{\Psi\}$ is compact, there exists a function $\Psi_0$ realizing this minimun, so that

$$W(\Phi, \Phi') = W(\Psi_0, \Phi, \Phi'),$$

although this function is not unique. We call such a translocation $\Psi_0$ a minimal translocation.

In what follows, we say that a translocation $\Psi$ from $x$ to $y$ is nonzero and write $x \to y$ if $\Psi(U_x, U_y) > 0$ for any neighborhoods $U_x$ and $U_y$ of the points $x$ and $y$. We call $\Psi$ a potential translocation if there exists a function $U(x)$ such that (1) $|U(x) - U(y)| \leq r(x, y)$, (2) $U(y) - U(x) = r(x, y)$ if $x \to y$.

Then the following theorem holds.

**Theorem.** *A translocation $\Psi$ is minimal if and only if it is potential.*

# Optimal transportation today

BASIC PROBLEM

To move mass from one place to another so as to:

- preserve volume, locally with respect to given densities or measures.
- minimize (or maximize) a cost.

# Monge-Kantorovich problem

DOMAINS:

$U, V \subset \mathbb{R}^n$, (or Riemannian manifold)
$U$: initial domain, $V$: target domain

DENSITIES:

$f, g \geq 0, \quad \in L^1(U), L^1(V)$ respectively

MASS BALANCE:

$$\int_U f = \int_V g$$

# Monge-Kantorovich problem

MASS PRESERVING MAPPINGS:

$T : U \to V$, Borel measurable,

$$\int_{T^{-1}(E)} f = \int_E g \qquad \forall \text{ Borel } E \subset V$$

$$\begin{aligned} \mathcal{T} &= \mathcal{T}(f, U; g, V) \\ &= \text{set of mass preserving mappings.} \end{aligned}$$

# Monge-Kantorovich problem

COST FUNCTION:

$$c : U \times V \to \mathbb{R}, \text{ continuous.}$$

COST FUNCTIONAL:

$$\mathcal{C} = \int_U c(x, Tx) f(x) dx$$

## The Problem

**Minimize** (or **maximize**) $\mathcal{C}$ over $\mathcal{T}$

# Remarks

1. More generally, densities can be replaced by measures $\mu$, $\nu$.
2. Kantorovich formulated relaxed version which permits *mass splitting*.
3. Modern parlance: $T$ pushes $\mu$ $(= fdx)$ forward to $\nu$ $(= gdy)$, with $T_{\#}\mu = \nu$.

# Applications

From Rachev and Ruschendorf:

*Mass Transportation Problems, 1998*

Econometrics

Functional analysis

Probability and statistics

Linear and stochastic programming

Differential geometry

Information theory

Cybernetics

Matrix theory

# Applications

More **recent** applications include:

Meteorology

Engineering design

Image processing

Traffic flow

Biological networks

Computing

Astrophysics

Mon. Not. R. Astron. Soc. 346, 501–524 (2003)

# Reconstruction of the early Universe as a convex optimization problem

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Accepted 2003 August 10. Received 2003 August 1; in original form 2003 April 10

**ABSTRACT**

We show that the deterministic past history of the Universe can be uniquely reconstructed from knowledge of the present mass density field, the latter being inferred from the three-dimensional distribution of luminous matter, assumed to be tracing the distribution of dark matter up to a known bias. Reconstruction ceases to be unique below those scales – a few Mpc – where multistreaming becomes significant. Above 6 $h^{-1}$ Mpc we propose and implement an effective Monge–Ampère–Kantorovich method of unique reconstruction. At such scales the Zel'dovich approximation is well satisfied and reconstruction becomes an instance of optimal mass transportation, a problem which goes back to Monge. After discretization into $N$ point masses one obtains an assignment problem that can be handled by effective algorithms with not more than $O(N^3)$ time complexity and reasonable CPU time requirements. Testing against $N$-body cosmological simulations gives over 60 per cent of exactly reconstructed points.

We apply several interrelated tools from optimization theory that were not used in cosmological reconstruction before, such as the Monge–Ampère equation, its relation to the mass transportation problem, the Kantorovich duality and the auction algorithm for optimal assignment. A self-contained discussion of relevant notions and techniques is provided.

**Key words:** hydrodynamics – cosmology: theory – early Universe – large-scale structure of Universe.

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 00, Number 0, Pages 000–000
S 0273-0979(XX)0000-0

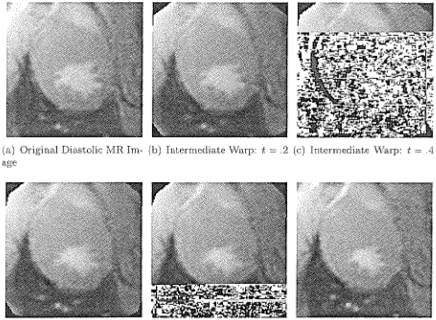
# MATHEMATICAL METHODS IN MEDICAL IMAGE PROCESSING

SIGURD ANGENENT, ERIC PICHON, AND ALLEN TANNENBAUM

⋮

5.2.3. *Optimal Warping.* Typically in elastic registration, one wants to see an explicit warping which smoothly deforms one image into the other. This can easily be done using the solution of the Monge–Kantorovich problem. Thus, we assume now that we have applied our gradient descent process as described above and that it has converged to the Monge–Kantorovich mapping $\tilde{u}_{MK}$.

⋮



(a) Original Diastolic MR Image (b) Intermediate Warp: $t = .2$ (c) Intermediate Warp: $t = .4$

(d) Intermediate Warp: $t = .6$ (e) Intermediate Warp: $t = .8$ (f) Original Systolic MR Image

FIGURE 5.2. Optimal Warping of Myocardium from Diastolic to Systolic in Cardiac Cycle. These static images become much more vivid when viewed as a short animation. (Available at http://www.bme.gatech.edu/groups/minerva/publications/papers/medicalBAMS2005.html).

# Primary Examples

1. ORIGINAL MONGE PROBLEM

$$c(x,y) = |x-y| \qquad x \in U, y \in V \qquad U, V \subset \mathbb{R}^n$$

(Monge 1781, $n = 2$ or 3, $f = g = 1$)

2. QUADRATIC COSTS

$$c(x,y) = \frac{1}{2}|x-y|^2 \qquad x \in U, y \in V \qquad U, V \subset \mathbb{R}^n$$

3. REFLECTOR ANTENNA

$$c(x,y) = -\log|x-y| \qquad x \in U, y \in V \qquad U, V \subset \mathbb{S}^n \rightsquigarrow \mathbb{R}^{n+1}$$

## Quadratic costs

$$c(x,y) = \frac{1}{2}|x-y|^2$$

This is equivalent to *maximizing*

$$c(x,y) = x.y$$

This problem was **solved** (uniquely a.e. $\{f > 0\}$) by Knott-Smith (1984), Brenier (1987) with solution

$$T = \nabla u$$

for convex potential $u$.

# Regularity - Caffarelli (1992,1996), Urbas (1997)

INTERIOR

$$V \text{ convex, } f, g \in C^\infty(U), C^\infty(V) \text{ resp.}$$

$$\inf f, g > 0 \Rightarrow u \in C^\infty(U)$$

GLOBAL

$$U, V \text{ uniformly convex, } f, g \in C^\infty(\overline{U}), C^\infty(\overline{V})$$

$$\inf f, g > 0 \Rightarrow u \in C^\infty(\overline{U})$$

# Monge-Ampère Equation

$$\det D^2 u = \frac{f}{g \circ Du}$$

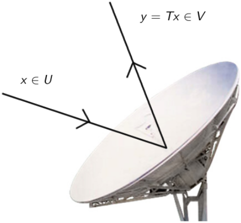
SECOND BOUNDARY VALUE PROBLEM

$$Tu(U) = V$$

solved by smooth diffeomorphism u

# Geometric Optics

Reflector antenna problem




$$U, V \in S^n \hookrightarrow \mathbb{R}^n$$

$$T_{\#} f dx = g dy$$

Reflecting surface:

$$\Gamma = \{x e^{-u(x)} | x \in U\}$$

$$Tx = x - \frac{2}{1 + |\nabla u|^2}(x + \nabla u)$$

## Geometric Optics

MONGE-AMPERE EQUATION:

$$\det\left\{\nabla^2 u + \nabla u \otimes \nabla u - \frac{1}{2}|\nabla u|^2 g_0 + \frac{1}{2} g_0\right\} = \left[\frac{1}{2}(1 + |\nabla u|^2)\right]^n f/g \circ T$$

INTERIOR REGULARITY: X-J WANG 1996, $n = 2$

OPTIMAL TRANSPORTATION FORMULATION: X-J Wang 2001

$$c(x, y) = -\log|x - y|$$

## Solution of Monge problem

SUDAKOV 1976 (Eng. trans. 1979)

- Measure decomposition
- 178 pages
- general norms: $c(x, y) = ||x - y||$.

EVANS-GANGBO 1999

- PDE approach, $p$-Laplacian, $p \to \infty$
- stronger asumptions on domains and densities.

## Solution of Monge problem

TRUDINGER-WANG 2001
CAFFARELLI-FELDMAN-MCCANN 2002

- simpler proofs
- approximation by strictly convex costs
- DRAMATIC DEVELOPMENT: Sudakov proof inadequate!
- restored by Ambrosio for original Monge cost in lectures (2000), then published in 2003.

$\Rightarrow$ Monge problem finally solved at the end of the twentieth century (T-Wang, Caffarelli-Feldman-McCann)

## General norms

AMBROSIO-KIRCHHEIM-PRATELLI (2004):

- Crystalline norms.

CHAMPION-DE PASCALE (2010):

- Different approach $\Rightarrow$ strictly convex norms.

CARAVENNA (2011):

- Restored Sudakov decomposition for strictly convex norms.

Monge-Sudakov problem, for strictly convex norms, finally solved at end of first decade !
(Champion-de Pascale, Caravenna)

- Extension to general convex norm (Champion - De Pascale 2011).

# Kantorovich potentials

KANTOROVICH DUAL PROBLEM:

Maximise

$$J(u,v) := \int_U fu + \int_V gv$$

over the set

$$K = \left\{ u, v \in C^0(\mathbb{R}^n) \;\middle|\; u(x) + v(y) \leq c(x,y) \;\; \forall x \in U, y \in V \right\}$$

with

$$J(u,v) \leq \mathcal{C}(T) \;\; \forall (u,v) \in K, T \in \mathcal{T}$$

# Kantorovich potentials

ASSUME: $c_x(x,\cdot)$ is one-to-one for all $x$.

$\Rightarrow \exists$ solutions $u, v$ , Lipschitz, with $u$ uniquely determined a.e. $\{f > 0\}$, such that

$$Tx = c_x^{-1}(x,\cdot)(Du)$$

solves associated Monge-Kantorovich problem.

Moreover $u$ and $v$ are dual, in particular,

$$v(y) = \inf_{x\in U}\{c(x,y) - u(x)\}, \qquad c - \text{transform}$$

$$u(x) = \inf_{y\in V}\{c(x,y) - v(y)\}, \qquad c^* - \text{transform}$$

Special case: $c(x,y) = c(x-y)$, strictly convex, Gangbo-McCann, Caffarelli (1996).

# Nonlinear partial differential equations

MONGE-AMPÈRE TYPE EQUATION:

$$\det\left[D^2 u - A(\cdot, Du)\right] = B(\cdot, Du)$$

OPTIMAL TRANSPORTATION: Assume $\det D^2_{x,y}(c) \neq 0$

$$A(x,p) \;=\; D_x^2 c(x, Y(x,p)) \;\;,\;\; Y(x,p) = c_x^{-1}(x,\cdot)(p)$$

$$B(x,p) \;=\; |\det D^2_{x,y} c| f/g \circ Y$$

# Nonlinear partial differential equations

## Monge-Ampère type equation:

$$\det\left[D^2u - A(\cdot, Du)\right] = B(\cdot, Du)$$

## Optimal transportation:

For convenience let $c, u \to -c, -u$.

Assume $\det D^2_{x,y}(c) \neq 0$

Then a Kantorovich potential $u \in C^2(U)$ satisfies MAE with

$$A(x,p) = D^2_x c(x, Y(x,p)) \quad , \quad Y(x,p) = c_x^{-1}(x,\cdot)(p)$$

$$B(x,p) = |\det D^2_{x,y} c| f/g \circ Y$$

## Nonlinear partial differential equations

Moreover since any potential $u$ is $c$-convex, i.e. $\forall x_0 \in U, \exists y_0 \in \overline{V}$ such that

$$u(x) - u(x_0) \geq c(x, y_0) - c(x_0, y_0)$$

we have

$$D^2 u \geq A(\cdot, Du)$$

if $u \in C^2(\Omega)$, i.e. MAE is degenerate elliptic w.r.t. $u$.

**Special case,** $c(x, y) = x.y$**,** $A \equiv 0$

- $c$-convex $\quad = \quad$ convex
- $D^2 u \geq 0 \quad = \quad$ locally convex

# The regularity problem

For what cost function and domains are there smooth (or diffeomorphism) solutions for smooth positive densities?

Villani, Topics in Optimal Transportation, 2003:

*"Without any doubt, the main open problem is to derive regularity estimates for more general transportation costs,... At the moment nothing is known concerning the smoothness of the solutions to these equations, beyond the regularity properties that automatically follow from c-concavity"*

# Condition A3 (Ma-T-Wang 2005)

So far...

A1: $c_x(x, \cdot)$ one-to-one for all $x$

A2: $\det c_{x,y} \neq 0$

Now...

A1*: $c_y(\cdot, y)$ one-to-one $\forall y$ (dual of A1)

A1, A2 $\Rightarrow A_{ij}(x,p) = c_{x_i x_j}(x, Y(x,p))$

(Recall that $c_x(x, Y(x,p)) = p$)

# Condition A3

**Define**

$$A_{ij}^{kl}(x,p) = D_{p_k p_l} A_{ij}(x,p)$$

This leads us to

A3 (A3w): $\quad A_{ij}^{kl}\xi_i\xi_j\eta_k\eta_l > 0, (\geq 0) \;\; \forall \xi,\eta \in \mathbb{R}^n, \;$ s.t $\xi.\eta = 0$

- $\mathcal{A} = [A_{ij}^{kl}]$ is a $2,2$ tensor in $x$ for each $y$.
- conditions A3, A3w are symmetric in $x$ and $y$.

$$A_{ij}^{kl} = (c_{ij,k'l'} - c^{r,s}c_{ij,s}c_{r,k'l'})c^{k',k}c^{l',l}$$

$$[c^{i,j}] = c_{x,y}^{-1}, \qquad c_{ij,\ldots kl} = \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\cdots\frac{\partial}{\partial y_k}\frac{\partial}{\partial y_l}\cdots c$$

## Generalized convexity

Let $c : \mathbb{R}^n \times \mathbb{R}^n$ be smooth, $c_x(x, \cdot)$ one-to-one for all $x$.

Then $u : \Omega \to \mathbb{R}$, $\Omega \in \mathbb{R}^n$, is $c$-convex in $\Omega$ if and only if $\forall x_0 \in \Omega$ $\exists\ y_0 \in \mathbb{R}^n$ such that $\forall x \in \Omega$

$$u(x) \geq u_0(x) := c(x, y_0) + u(x_0) - c(x_0, y_0)$$

$u \in C^2(\Omega)$ is locally $c$-convex in $\Omega$ if and only if

$$D^2 u \geq D_x^2 c(\cdot, Y(\cdot, Du)), \qquad Y(x,p) = c_x(x, \cdot)^{-1}(p)$$

**Q1** For what $c$ and domains $\Omega$ does local $c$-convexity imply (global) $c$-convexity?

## Generalized convexity

NORMAL MAPPING:

$$
\begin{aligned}
Tu(x_0) &= \left\{ y_0 \in \mathbb{R}^n | u \geq u_0 \in \Omega \right\} \\
&\subset Y(x_0, \partial u(x_0)), \quad (= \text{ a.e. })
\end{aligned}
$$

**Q2** For what $c$ does $Tu = Y(\cdot, \partial u)$?

CONTACT SET:

$$
\Gamma = \Gamma(x_0, y_0) = \{x \in \Omega | u(x) = u_0(x)\}, \quad y_0 \in Tu(x_0)
$$

**Q3** For what $c$ does it follow that $\Gamma$ is connected?

# Domain convexity

- $U$ is convex w.r.t $E \subset \mathbb{R}^n \Longleftrightarrow c_y(\cdot, y)(U)$ is convex in $\mathbb{R}^n, \forall y \in E$.
- $U$ is uniformly $c$-convex w.r.t $E \Longleftrightarrow c_y(\cdot, y)(U)$ is uniformly convex w.r.t $y \in E$.
- $c^*(x, y) = c(y, x) \Rightarrow$ analogous definitions for $V$.
- $c(x, y) = x.y \Rightarrow$ usual convexity, $c_y = I$
- Small balls are uniformly convex
- Invariant under coordinate changes

# Specific examples

1. Power costs

- $$c(x,y) = \pm \begin{cases} \frac{1}{m}|x-y|^m & , m \neq 0,1 \\ \log|x-y| & , m = 0 \end{cases}$$

$$A(x,p) = A(p) = \mp \left\{ |p|^{\frac{m-2}{m-1}} I + (m-2)|p|^{-\frac{m}{m-1}} p \otimes p \right\}$$

$+$ case: A3w only for $m = 2$

$-$ case: A3w for $-2 \leq m < 1$

A3 for $-2 < m < 1$

# Specific examples

1. POWER COSTS (CTD.)

- vector field $$Y(x,p) = x \pm |p|^{\frac{2-m}{m-1}} p$$
- $c(x,y) = \sum |x_i - y_i|^{m_i}, m_i \geq 2$ satisfies A3w.

# Specific examples

2. Graph distances

$M_f, M_g \subset \mathbb{R}^{n+1}$, graphs of $f, g \in C^2(U), C^2(V)$ resp.

$$Df(x).Dg(y) > -1 \qquad \forall x \in U, y \in V$$

$$c(x,y) = \frac{1}{2}|\hat{x} - \hat{y}|^2$$

where $\hat{x} = (x, x_{n+1}) \in M_f$, $\hat{y} = (y, y_{n+1}) \in M_g$, satisfies

$$\begin{cases} A3w & \text{if } f,g \text{ convex} \\ A3 & \text{if } f,g \text{ uniformly convex} \end{cases}$$

# Specific examples

2. GRAPH DISTANCES

**Examples:**

- $f = \sqrt{1 + |x|^2}, \quad U \subset \mathbb{R}^n$
- $f = -\sqrt{1 - |x|^2}, \quad U \subset B_{1/\sqrt{2}}(0)$
- $f = \epsilon |x|^2 \Rightarrow$ A3 approximation to $c(x, y) = \frac{1}{2}|x - y|^2$.
- Level sets of $f$ are $c$-convex.

# Specific examples

3. $c(x, y) = \sqrt{1 + |x - y|^2}$

- $A(x, p) = A(p) = -\sqrt{1 - |p|^2}(I - p \otimes p)$
- satisfies A3
- vector field $Y(x, p) = x + p/\sqrt{1 - |p|^2}$.
- Lorentzian curvature
- $c_\epsilon(x, y) = \sqrt{\epsilon^2 + |x - y|^2} \quad \to$ Monge cost $|x - y|$
- $U$ is $c$-convex w.r.t. $V$ if $V \subset U$.

# Specific examples

4. $c(x,y) = \sqrt{1 - |x-y|^2}$

- $A(x,p) = A(p) = \sqrt{1+|p|^2}(I + p \otimes p)$
- satisfies A3
- vector field $Y(x,p) = x - p/\sqrt{1+|p|^2}$.
- Euclidean curvature

Note: $c \to -c$ in computing $Y$ and $A$

# Interior regularity

THEOREM (Ma-T-Wang 2005, correction, T-Wang 2009)

- Cost function $c \in C^\infty$ satisfies A1, A1*, A2, A3.
- Domain $V$ is $c^*$-convex w.r.t. $U$.

Densities $f, g \in C^\infty(U), C^\infty(V)$ resp. $\inf f, g > 0$

$\Rightarrow$ optimal mapping $T \in [C^\infty(U)]^n$.

# Interior regularity

THEOREM (Loeper 2009)

Densities $f \in L^p(U), p > n, \inf g > 0$

$\Rightarrow T \in [C^{0,\alpha}]^n$ for some $\alpha > 0$.

THEOREM (Liu, improvement 2009)

$$f \in L^p(\Omega), p > (n+1)/2 \Rightarrow \alpha = \frac{\beta(n+1)}{2n^2 + \beta(n-1)}, \beta = 1 - \frac{n+1}{2p}$$

**(sharp)**

# Interior regularity

THEOREM (Liu-T-Wang 2009)

$f, g \in C^{0,\alpha}(U), C^{0,\alpha}(V), 0 < \alpha < 1$

$\Rightarrow T \in [C^{1,\alpha}(U)]^n, 0 < \alpha < 1$

THEOREM (Figalli-Kim-McCann, preprint 2011)

- Cost function $c \in C^\infty$ satisfies A1, A1*, A2, A3w.
- Domain $V$ is uniformly $c^*$-convex w.r.t. $U$.
- $f, g \in L^\infty$, $\inf 1/f, 1/g > 0$

$\Rightarrow T \in [C^{0,\alpha}]^n$ for some $\alpha > 0$

## Boundary regularity

THEOREM (T-Wang 2009, T 2013)

- Cost function $c \in C^\infty$ satisfies A1, A2, A3w
- Domains $U, V \in C^\infty$, uniformly $c, c^*$ convex
- Densities $f, g \in C^\infty(\overline{U}), C^\infty(\overline{V})$ resp. $\inf f, g > 0$

$\Rightarrow \exists$ a unique (a.e.) optimal diffeomorphism $T \in [C^\infty(\overline{U})]^n$ given by

$$T = Y(\cdot, Du)$$

where $u \in C^\infty(\overline{U})$ is elliptic solution of PDE

$$\det[D^2 u - D_x^2 c(\cdot, Y(\cdot, Du))] = (\det c_{x,y}) f/g \circ Y$$

# Transportation in Riemannian manifolds

1. Extrinsic costs

$$c : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}, M \rightsquigarrow \mathbb{R}^{n+1}$$

**Examples:**

- Light reflector problem:

$$M = S^n, c(x, y) = -\log |x - y|$$

  satisfies A3.

- Quadratic cost

$$M = S^n, c(x, y) = \frac{1}{2}|x - y|^2$$

  related to graph example,
  satisfies A3 for $x, y > 0$.

## Transportation in Riemannian manifolds

2. Intrinsic costs

$$c(x, y) = \frac{1}{2}[d(x, y)]^2$$

where $d(x, y)$ is the geodesic distance between $x$ and $y$.

- A3w $\Rightarrow$ sectional curvatures $\geq 0$ (Loeper 2009)
  $\Rightarrow$ no regularity in hyperbolic manifolds.

For sphere $M = S^n$, satisfies A3 (Loeper 2009)

Not true for general ellipsoids (Figalli -Rifford-Villani 2010)

Recent developments, including relationship with cut locus: Kim-McCann 2012, Delanoe-Ge 2010, 2011, Loeper-Villani 2010, Figalli-Rifford-Villani 2011, 2012.

## Convexity theory

Assume $c$ satisfies A1, A1*, A2, A3w. $\Rightarrow$

1. $\Omega$ $c$-convex w.r.t $Tu(\Omega)$, $u$ locally $c$-convex, $\in C^2(\Omega) \Rightarrow u$ (globally) $c$-convex (T-Wang 2009).
2. Normal mapping $T = Y(\cdot, \partial u)$, $\Longleftrightarrow$ by duality
3. $\Omega$ $c$-convex w.r.t $y_0 \Rightarrow$ contact set $\Gamma = \Gamma(x_0, y_0)$ is connected $\forall x_0 \in \Omega$, (Loeper 2009, T-Wang 2009, Kim-McCann 2010)
4. Sharpness: Target $V$ $c^*$-convex necessary for optimal map $T$ to be cts (Ma-T-Wang 2005).
5. Sharpness: A3w necessary for optimal map $T$ to be cts, as well as for 2 and 3 (Loeper 2009).