ABSTRACT

Picard-Vessiot extensions for real fields

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Abstract

Picard-Vessiot theory can be described as Galois theory of linear differential equations. This theory is a generalization of classical Galois theory for polynomial equations to homogeneous linear differential equations. Picard-Vessiot theory is due to E. Picard and E. Vessiot and in rigorous form to E. Kolchin, who built on the work of J.F. Ritt in differential algebra. It was made more accessible by the book of I. Kaplansky.

Picard-Vessiot theory has been built under the hypothesis that the field of constants C_K of the differential field K, over which the differential equation is defined, is algebraically closed. In this case, one obtains existence and uniqueness, up to K-differential isomorphisms, of the Picard-Vessiot extension of the differential equation and that the differential Galois group of the differential equation, defined as the group of K-differential automorphisms of its Picard-Vessiot extension, is a linear algebraic group over C_K. It is worth considering whether the condition C_K algebraically closed can be weakened. In particular, the case of real fields is interesting.

Many interesting and significant results concerning differential algebra for real fields can be found in papers by M. Singer, T. Dyckerhoff, T. Grill, M. Knebusch, M. Tressl. But an existence theorem for Picard-Vessiot extension even in the case $C_K = \mathbb{R}$ has never been proved. The reason for that might be a commentary of Armand Borel, which can be found in his article contained in *Selected Works of Ellis Kolchin* (see [1]). Borel wrote about the proof by Kolchin of the existence theorem of Picard-Vessiot theory:

This is under our standing assumption that C_F is algebraically closed (of char. 0). If not, then Seidenberg has produced an equation such that $C_E \neq C_F$ for all differential field extensions E generated over F by a fundamental set of solutions of that equation.

But if we take a close look at the example of Seidenberg mentioned above (see chapter 4.1 of this work) we will see that the base field F is *not a real field.*

A Picard-Vessiot extension is a differential field extension $K \subset L$ such that there exists a following homogeneous linear ordinary differential equation $\mathcal{L}(Y) := Y^{(n)} + a_{n-1}Y^{(n-1)} + \ldots + a_1Y' + a_0Y = 0$ with coefficients in K having a fundamental set of solutions $y_1, \ldots, y_n$ in L which differentially generates L over K and moreover the fields of constants of K and L coincide.

Kolchin proved the existence of a Picard-Vessiot extension for a given homogeneous linear ordinary differential equation over a differential field with algebraically closed field of constants. In one of his papers Kolchin indicates that the difficulty in proving the existence of a Picard-Vessiot extension for a given equation lies in proving that such an extension brings in no new constants. He also comments that this problem was formulated earlier in 1933 by Reinhold Baer in his commentary which appears in Felix Klein's book *Vorlesungen über hypergeometrische Funktion*.

The main result of this work is proving that for a given homogeneous linear ordinary differential equation defined over a real differential field K with field of constants a real closed field F there exists a Picard-Vessiot extension which is also a real field.

Our work is organised as follows:

In chapter 1 we introduce the notions of differential ring and differential field and we study some preliminary facts of differential algebra. We also prove Ritt-Raudenbusch Basis Theorem which is an equivalent of Hilbert's Basis Theorem for radical differential ideals and a differential version of primary decomposition theorem. We introduce the very useful notion of Taylor morphism and present a differential version of the primitive element theorem.

In chapter 2 we give a brief introduction to the theory of real and real closed fields. We present Tarski-Seidenberg Principle and its consequences. We also study some other results needed in further chapters, like Skolem-Löwenheim Theorem.

In chapter 3 we introduce the basic definitions and state some essential results of Picard-Vessiot theory. We consider differential fields of characteristic zero with algebraically closed fields of constants. Here we present the theorem on the existence and uniqueness of Picard-Vessiot extension in such case and the Fundamental Theorem of Picard-Vessiot Theory.

The last chapter contains our new results. We study homogeneous linear ordinary differential equations defined over a real differential field K differentially finitely generated over a real closed field F considered as a differential field with trivial derivation, with field of constants equal to F. In this case we construct a real Picard-Vessiot extension. We obtain the general result by applying the Kuratowski-Zorn lemma. Finally, we give a short commentary on the Fundamental Theorem of Picard-Vessiot Theory in the case considered.

[1] A. Borel, *Algebraic Groups and Galois Theory in the Works of Ellis Kolchin* in *Selected works of Ellis Kolchin with commentary*, H. Bass, A. Buium and P.J. Cassidy, eds. American Mathematical Society, Providence, RI, pp. 505-525, 1999.