MARKOV'S INEQUALITY IN THE COMPLEX PLANE

Raimondo Eggink

Jagiellonian University, Cracow

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PREFACE

In the spring of 1993 prof. L.P. Bos and prof. P.D. Milman circulated a preprint titled "A Geometric Interpretation and the Equality of Exponents in Markov and Gagliardo-Nirenberg (Sobolev) Type Inequalities for Singular Compact Domains", in which they proved the equivalence of a local and global Markov inequality for polynomials on compact sets in $\mathbb{R}^N$, $N \in \mathbb{N}$. They were able to achieve this remarkable result by defining and ultimately proving the equivalence of several Sobolev-type inequalities and extension properties for smooth functions.

The assignment for my Master's thesis, written in 1994 under the supervision of prof. J. Siciak, was to attempt to generalize the work by L.P. Bos and P.D. Milman for the case of compact sets in the complex plane. At that time I succeeded only to a very limited extent, but some (partial) results and ideas proved to be useful in subsequent research of the problem. The main difficulty lay in finding suitable generalizations of the respective properties, the lack of an equivalent of the classic Jackson theorems in the complex plane and obviously its complex structure.

Jointly with dr. L. Białas-Cież we have continued this research during the past eight years. We published one joint article titled "L-regularity of Markov Sets and of m-Perfect Sets in the Complex Plane" in the journal Constructive Approximation [Białas-Eggink 1] and we submitted for publication our second article titled "Equivalence of the Local Markov Inequality and a Sobolev Type Inequality in the Complex Plane" [Białas-Eggink 2]. In this dissertation I present and expand upon the results of these two articles, as well as subsequent research, which will be part of the source material for our next two articles with the working titles "Łojasiewicz-Siciak Inequality of Green's Function and a Version of Jackson's Theorem in the Complex Plane" and "Equivalence of the Global and Local Markov Inequalities in the Complex Plane".

Our main result is that the entire proof by L.P. Bos and P.D. Milman can be generalized to the case of a compact set in the complex plane, provided however that for this set we have an additional assumption to compensate for the lack of Jackson's theorem. Indeed we know now that without such an additional assumption, the global Markov inequality does not imply the local Markov inequality in the complex plane. We are still searching for the weakest possible additional assumption needed to obtain equivalence, but in this dissertation I propose a sufficient assumption under the name Jackson Property. Furthermore we have only just started to study the complex structure of sets without this property.

The contents of this dissertation can be outlined as follows. Chapter 1 contains a brief reminder of the preliminaries needed to understand the rest of the text, mostly by reference to the work on related subjects by well-known authors. In chapter 2 several versions of local Markov inequalities are studied, mostly for the sake of completeness and comparision with the version introduced by L.P. Bos and P.D. Milman, presented here in chapter 3.

Chapter 4 deals with the geometric structure and logarithmic capacity of sets admitting the Local Markov Property. It is proven that these sets are (m, ∞)-perfect, which was conjectured earlier in [Eggink], and this in turn yields L-regularity.

Chapters 5, 6 and 7 give a complete proof of the generalization of [Bos-Milman, theorem A], which asserted the equivalence of the Local Markov Property and Sobolev-type inequalities in different norms. Particular care is taken to formulate the last of the Sobolev properties in such a way to make it appear as weak as possible, so that it can easily be deduced from an extension property.

In chapter 8 the Jackson Property is introduced together with some straightforward examples. More importantly, a refinement of Runge's theorem allows to link this property with the behaviour of Siciak's extremal function. Accordingly, it is proved that sets admitting the Hölder Continuity Property as well as the Łojasiewicz-Siciak inequality, i.e. estimates for the extremal function from above and below, respectively, admit also the Jackson Property.

Chapter 9 generalizes, to the extent possible, the original extention theorem for sets admitting the Global Markov Inequality, due to prof. W. Pleśniak for compact sets in $\mathbb{R}^N$ and based on his earlier joint work with prof. W. Pawłucki [Pleśniak 1]. It served as the starting point for the extension property used by L.P. Bos and P.D. Milman, which is redefined here in chapter 10 for compact sets in the complex plane. This chapter culminates with our main result announced above, which corresponds to [Bos-Milman, theorem B]. On the other hand it also gives an example, brought to our attention by J. Siciak, showing that the Jackson Property is not the weakest possible assumption needed to derive the Local Markov Property from the Global Markov Inequality.

Finally chapter 11 discusses a handful of open problems, which are the subject of our ongoing research. For the convenience of the reader, at the end of this dissertation there is a graphical overview of its results.

It should be noted that there is a rich literature concerning many different versions and aspects of Markov-type polynomial inequalities and related topics, see e.g. [Pleśniak 3] or [Pleśniak 5] and [Frerick] for excellent surveys. Most authors however are preoccupied with sets in $\mathbb{R}^N$. In the complex plane, the notion of a compact set that is e.g. uniformly polynomially cuspidal, Whitney p-regular or semi-analytic becomes trivial, because every connected set admits a local and global Markov inequality. Therefore we are really most interested in sets that are totally disconnected or otherwise highly irregular.

No attempt whatsoever has been made to generalize any of these results for the multivariate case in $\mathbb{C}^N$, where $N \in \mathbb{N} \setminus \{1\}$, nor for L^{p} norms. On the other hand a lot of attention was paid to producing self-contained if not simplistic proofs and optimizing the main coefficients, which naturally does not imply, that they cannot be improved further. Any shortcomings in this dissertation are solely my responsibility.

Foremost I wish to extend my special gratitude to L. Białas-Cież for our fruitful mathematical co-operation throughout the years, which also included a significant amount of crucial practical support. I am much obliged to my academic advisor and friend prof. A. Edigarian for motivating me to finish this dissertation. Furthermore I take the opportunity to thank all my former teachers and fellow students at the Institute of Mathematics of the Jagiellonian University in Cracow for a wonderful educational experience. Finally this work would not have been possible without the patient support of my loving wive Elżbieta and our sons Mateusz, Ryszard, Aleksander and Przemysław.

Raimondo Eggink

CHAPTER I

PRELIMINARIES

The reader is assumed to be acquainted with basic academic courses in real and complex analysis like e.g. [Leja 2], [Łojasiewicz] or [Rudin 2], as well as potential theory, e.g. [Tsuji 2] or [Ransford]. Furthermore we will use the following definitions and facts, all well known to specialists in the field.

In the year 1889 A.A. Markov proved his famous polynomial inequality:

THEOREM 1.1 [Markov].

$$\forall n \in \mathbb{N} \quad \forall p \in \mathcal{P}_n \quad : \quad \|p'\|_{[-1,1]} \leq n^2 \cdot \|p\|_{[-1,1]}.$$

Equally well known is S.N. Bernstein's inequality for trigonometric polynomials [Bernstein 2, chapter 1], which translates to a Markov inequality for the closed unit ball in the complex plane.

THEOREM 1.2.

$$\forall n \in \mathbb{N} \quad \forall p \in \mathcal{P}_n \quad : \quad \|p'\|_{B(0,1)} \leq n \cdot \|p\|_{B(0,1)}.$$

Comprehensive proofs of these theorems can be found in the handbooks [Pleśniak 4, chapter 11], [DeVore, chapter 4 §1], [Rahman-Schmeisser, chapter 1] or [Cheney, chapter 3 section 7].

DEFINITION 1.3. A compact set $E \subset\subset \mathbb{K}$, where $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$, admits the Global Markov Inequality GMI(k) where $k \geq 1$, if

$$\exists M \geq 1 \quad \forall n \in \mathbb{N} \quad \forall p \in \mathcal{P}_n(\mathbb{K}) \quad : \quad \|p'\|_E \leq M \cdot n^k \cdot \|p\|_E.$$

We will write that the set E admits GMI if it admits GMI(k) for some $k \geq 1$. We employ the usual supremum norm, i.e. $\|p\|_E := \sup_{z \in E} |p(z)|$.

REMARK 1.4. Note that if a compact set $E \subset\subset \mathbb{R}$ admits GMI(k) for real polynomials, then as a set on the complex plane it admits GMI(k) for complex polynomials too.

Furthermore the property GMI(k) is invariant under a linear change of the variable (except for the constant M, of course).

DEFINITION 1.5 [Leja 2, chapter 11; Fekete]. For a compact set $E \subset\subset \mathbb{C}$ and $n \in \mathbb{N}$ we define a set of n Fekete extremal points, denoted $\left\{\zeta_1^{(n)}, \ldots, \zeta_n^{(n)}\right\} \subset E$. For $z_1, \ldots, z_n \in \mathbb{C}$ we put

$$V(z_1, \ldots, z_n) := \prod_{1 \leq \mu < \nu \leq n} (z_\nu - z_\mu)$$

and subsequently we find a set of n points $\left\{\zeta_1^{(n)}, \ldots, \zeta_n^{(n)}\right\} \subset E$ such that

$$\left| V\left(\zeta_1^{(n)}, \ldots, \zeta_n^{(n)}\right) \right| = \max \left\{ |V(z_1, \ldots, z_n)| \ : \ z_1, \ldots, z_n \in E \right\}.$$

Obviously such a set of points does not need to be unique, but its existence is guaranteed by the compactness of the set $E^n = \overbrace{E \times \ldots \times E}^{n \text{ times}}$.

REMARK 1.6. For a given set of n Fekete extremal points $\left\{\zeta_1^{(n)}, \ldots, \zeta_n^{(n)}\right\} \subset E$, where $n \in \mathbb{N} \setminus \{1\}$, we denote

$$d_n(E) := \left(\prod_{1 \leq \mu < \nu \leq n} |\zeta_\mu^{(n)} - \zeta_\nu^{(n)}| \right)^{1/\binom{n}{2}}.$$

From the papers by M. Fekete and F. Leja concerning the transfinite diameter [Tsuji 2, chapter III §5; Leja 2, chapter 11; Fekete], it is now commonly known that

$$\lim_{n \to \infty} d_n(E) = \operatorname{cap} E.$$

Definition 1.7 [Leja 2, chapter 11]. For a compact set $E \subset\subset \mathbb{C}$, any subset of $n+1$ distinct interpolation knots $\left\{\zeta_0^{(n)}, \ldots, \zeta_n^{(n)}\right\} \subset E$, where $n \in \mathbb{N}$, and a function $f \in \mathcal{C}(E)$ we define the following Lagrange interpolation polynomial of degree n:

$$L_n f(z) := \sum_{\mu=0}^{n} f\left(\zeta_\mu^{(n)}\right) \cdot L_{n,\mu}\left(z; \zeta_0^{(n)}, \ldots, \zeta_n^{(n)}\right),$$

where for $\mu = 0, \ldots, n$ we put

$$L_{n,\mu}\left(z; \zeta_0^{(n)}, \ldots, \zeta_n^{(n)}\right) := \prod_{\substack{\nu=0,\ldots,n \\ \nu \neq \mu}} \frac{z - \zeta_\nu^{(n)}}{\zeta_\mu^{(n)} - \zeta_\nu^{(n)}}.$$

Remark 1.8. We see that for all $\mu, \nu = 0, \ldots, n$ we have

$$L_{n,\mu}\left(\zeta_\nu^{(n)}; \zeta_0^{(n)}, \ldots, \zeta_n^{(n)}\right) = \begin{cases} 0 & \text{if } \mu \neq \nu, \\ 1 & \text{if } \mu = \nu, \end{cases}$$

and thus $L_n f\left(\zeta_\nu^{(n)}\right) = f\left(\zeta_\nu^{(n)}\right)$.

If we use Fekete extremal points as the interpolation knots, then for all $z \in E$ and $\mu = 0, \ldots, n$ we have $\left|L_{n,\mu}\left(z; \zeta_0^{(n)}, \ldots, \zeta_n^{(n)}\right)\right| \leq 1$ and therefore

$$\|L_n f\|_E \leq \sum_{\mu=0}^{n} \|f\|_E \cdot \left\|L_{n,\mu}\left(\,\cdot\,; \zeta_0^{(n)}, \ldots, \zeta_n^{(n)}\right)\right\|_E = (n+1) \cdot \|f\|_E.$$

Furthermore it is obvious that L_n is a linear operator maintaining polynomials of degree n or less. Consequently if we assume $p \in \mathcal{P}_n$ to be the polynomial of best approximation, i.e. $\|f - p\|_E = \operatorname{dist}_E(f, \mathcal{P}_n) := \inf_{q \in \mathcal{P}_n} \|f - q\|_E$, then we see that

$$\|L_n f - p\|_E = \|L_n f - L_n(p_{|E})\|_E = \|L_n(f - p_{|E})\|_E \leq (n+1) \cdot \|f - p\|_E = (n+1) \cdot \operatorname{dist}_E(f, \mathcal{P}_n),$$

$$\|f - L_n f\|_E = \|(f-p) - (L_n f - p)\|_E \leq \|f - p\|_E + \|L_n f - p\|_E \leq (n+2) \cdot \operatorname{dist}_E(f, \mathcal{P}_n).$$

This demonstrates that the Lagrange interpolation polynomials with knots in Fekete extremal points have good approximation qualities.

Definition 1.9 [Siciak 1; cf. Leja 2, chapter 11]. For a compact set $E \subset\subset \mathbb{C}$ we define Siciak's extremal function with respect to holomorphic polynomials

$$\Phi_E(z) := \limsup_{n \to \infty} \sqrt[n]{\Phi_n(z)} \quad \text{for } z \in \mathbb{C}$$

where

$$\Phi_n(z) := \sup\{|p(z)| \;:\; p \in \mathcal{P}_n \;,\; \|p\|_E \leq 1\}.$$

Note that we have equivalently

$$\Phi_E(z) = \sup\left\{|p(z)|^{1/\deg p} \;:\; p \in \mathcal{P} \;,\; \deg p \geq 1 \;,\; \|p\|_E \leq 1\right\}.$$

Definition 1.10. For a compact set $E \subset\subset \mathbb{C}$ we define its polynomial hull

$$\hat{E} := \{z \in \mathbb{C} \;:\; \forall p \in \mathcal{P} \quad |p(z)| \leq \|p\|_E\} = \{z \in \mathbb{C} \;:\; \Phi_E(z) = 1\}.$$

If $E = \hat{E}$ then we say that the set E is polynomially convex. Note that by the maximum principle for holomorphic functions, the complement of a polynomially convex set is simply connected.

We summarize below some important properties of the extremal function. Their proofs can be found in the numerous papers of its creators, e.g. [Siciak 1], [Leja 2] and [Leja 1].

THEOREM 1.11. *Let $E \subset\subset \mathbb{C}$ be a compact set.*
(a) If $\operatorname{cap} E = 0$ then the set E is called polar and furthermore we have

$$\Phi_E(z) = \begin{cases} 1 & \text{if } z \in E, \\ +\infty & \text{if } z \notin E. \end{cases}$$

(b) If $\operatorname{cap} E > 0$ then we have

$$\text{(i)} \qquad \forall z \in \mathbb{C} \setminus \hat{E} \quad : \quad \Phi_E(z) = e^{g_E(z)},$$

where g_E is Green's function of the set $\mathbb{C} \setminus \hat{E}$ with its pole at infinity;

$$\text{(ii)} \qquad \lim_{z\to\infty} \frac{|z|}{\Phi_E(z)} = \operatorname{cap} E;$$

$$\text{(iii)} \qquad \forall p \in \mathcal{P} \quad \forall z \in \mathbb{C} \quad : \quad |p(z)| \leq \big(\Phi_E(z)\big)^{\deg p} \cdot \|p\|_E,$$

which is called the Bernstein-Walsh-Siciak inequality.
(c) If the set E is connected, then we have

$$\forall z \in \mathbb{C} \setminus \hat{E} \quad : \quad \Phi_E(z) = |\psi(z)|\,,$$

where $\psi : \hat{\mathbb{C}} \setminus \hat{E} \to \hat{\mathbb{C}} \setminus B(0,1)$ is a conformal mapping such that $\psi(\infty) = \infty$.

DEFINITION 1.12. A compact set $E \subset\subset \mathbb{C}$ is called L-regular if its extremal function Φ_E is continuous on the entire complex plane.

DEFINITION 1.13. For a compact set $E \subset\subset \mathbb{C}$ we define a closed neighbourhood with radius $\delta > 0$:

$$E_\delta := \{z \in \mathbb{C} \ : \ \operatorname{dist}(z, E) \leq \delta\}\,.$$

DEFINITION 1.14. A compact set $E \subset\subset \mathbb{C}$ admits the Hölder Continuity Property HCP(k), where $k \geq 1$, if

$$\exists M \geq 1 \quad \forall z \in E_1 \quad : \quad \Phi_E(z) \leq 1 + M \cdot \operatorname{dist}(z, E)^{1/k}.$$

We will write that the set E admits HCP if it admits HCP(k) for some $k \geq 1$.

DEFINITION 1.15. A compact set $E \subset\subset \mathbb{C}$ admits the Łojasiewicz-Siciak inequality ŁS(s), where $s \geq 1$, if

$$\exists M > 0 \quad \forall z \in E_1 \quad : \quad \Phi_E(z) \geq 1 + M \cdot \operatorname{dist}(z, E)^s.$$

We will write that the set E admits ŁS if it admits ŁS(s) for some $s \geq 1$.

REMARK 1.16. Note that both properties HCP and ŁS can be defined equivalently in terms of Green's function instead of the extremal function, because for arbitrary $t > 0$ we have

$$1 + g_E(z) \leq e^{g_E(z)} = \Phi_E(z) \leq 1 + \frac{e^t - 1}{t} \cdot g_E(z)$$

for all $z \in \mathbb{C} \setminus \hat{E}$, such that $0 \leq g_E(z) \leq t$.

COROLLARY 1.17. *For a compact set $E \subset\subset \mathbb{C}$ we have the following implications:*

$$\text{HCP} \implies \textit{L-regularity} \implies \operatorname{cap} E > 0.$$

PROPOSITION 1.18 [Pleśniak 1, theorem 3.3.ii]. *For a compact set $E \subset\subset \mathbb{C}$ and $k \geq 1$ the following conditions are equivalent:*

(i) $$\mathrm{GMI}(k) \quad \text{i.e.} \quad \exists M \geq 1 \quad \forall n \in \mathbb{N} \quad \forall p \in \mathcal{P}_n \quad : \quad \|p'\|_E \leq M \cdot n^k \cdot \|p\|_E,$$

(ii) $$\exists \widetilde{M} \geq 1 \quad \forall n \in \mathbb{N} \quad \forall p \in \mathcal{P}_n \quad : \quad \|p\|_{E_{1/n^k}} \leq \widetilde{M} \cdot \|p\|_E.$$

PROOF. (i)$\Longrightarrow$(ii) We put $\widetilde{M} := e^M$. Fix arbitrary $n \in \mathbb{N}$ and $p \in \mathcal{P}_n$ so that for any point $z_0 \in \mathbb{C}$ we have $p(z) = \sum_{j=0}^{n} \frac{p^{(j)}(z_0)}{j!} \cdot (z - z_0)^j$. Specifically, if for an arbitrary point $z \in E_{1/n^k}$ we select a point $z_0 \in E$ such that $|z - z_0| = \mathrm{dist}(z, E) \leq \frac{1}{n^k}$, then we obtain

$$|p(z)| \leq \sum_{j=0}^{n} \frac{\|p^{(j)}\|_E}{j!} \cdot \mathrm{dist}(z, E)^j.$$

By iteration of GMI we see that $\|p^{(j)}\|_E \leq M^j \cdot n^{k \cdot j} \cdot \|p\|_E$ and therefore

$$|p(z)| \leq \sum_{j=0}^{n} \frac{M^j \cdot n^{k \cdot j} \cdot \|p\|_E}{j!} \cdot \frac{1}{n^{k \cdot j}} \leq e^M \cdot \|p\|_E = \widetilde{M} \cdot \|p\|_E.$$

(ii)$\Longrightarrow$(i) We put $M := \widetilde{M}$. Fix arbitrary $n \in \mathbb{N}$, $p \in \mathcal{P}_n$ and $z \in E$. Cauchy's integral formula tells us that

$$p'(z) = \frac{1}{2\pi i} \cdot \int_{\partial B(z, 1/n^k)} \frac{p(\zeta)}{(\zeta - z)^2} \cdot d\zeta$$

and thus

$$|p'(z)| \leq \frac{1}{2\pi} \cdot \int_{\partial B(z, 1/n^k)} \frac{\|p\|_{E_{1/n^k}}}{(1/n^k)^2} \cdot |d\zeta| = \frac{\|p\|_{E_{1/n^k}}}{1/n^k} \leq \widetilde{M} \cdot n^k \cdot \|p\|_E = M \cdot n^k \cdot \|p\|_E. \quad \square$$

PROPOSITION 1.19. *If a compact set $E \subset\subset \mathbb{C}$ admits* HCP(k)*, where $k \geq 1$, then it also admits* GMI(k).

PROOF. By the assumption, if $\mathrm{dist}(z, E) \leq 1$ then $\Phi_E(z) \leq 1 + M \cdot \mathrm{dist}(z, E)^{1/k}$. Fix arbitrary $n \in \mathbb{N}$, $p \in \mathcal{P}_n$ and $z \in E_{1/n^k}$. Then by the Bernstein-Walsh-Siciak inequality we obtain

$$|p(z)| \leq \left(\Phi_E(z)\right)^n \cdot \|p\|_E \leq \left(1 + M \cdot \mathrm{dist}(z, E)^{1/k}\right)^n \cdot \|p\|_E \leq$$
$$\leq \left(1 + M \cdot \left(\frac{1}{n^k}\right)^{1/k}\right)^n \cdot \|p\|_E = \left(1 + \frac{M}{n}\right)^n \cdot \|p\|_E \leq e^M \cdot \|p\|_E.$$

So we see that the set E admits condition (ii) of proposition 1.18, where $\widetilde{M} := e^M$. $\square$

REMARK 1.20. It is widely known that every continuum in the complex plane admits HCP(2) and thus GMI(2). L. Białas-Cież and A. Volberg proved in [Białas-Volberg, proposition 5.1] that the Cantor ternary set admits HCP, with a coefficient that is still the subject of ongoing research.

PROPOSITION 1.21. *If a polynomially convex compact set $E \subset\subset \mathbb{C}$ admits* GMI*, then it is perfect.*

PROOF. By the assumption we have

$$\exists M \geq 1 \quad \exists k \geq 1 \quad \forall n \in \mathbb{N} \quad \forall p \in \mathcal{P}_n(\mathbb{C}) \quad : \quad \|p'\|_E \leq M \cdot n^k \cdot \|p\|_E.$$

Let's assume to the contrary that the set E is not perfect and therefore we can find an isolated point $z_0 \in E$, so that the set $E \setminus \{z_0\}$ is compact, polynomially convex and $z_0 \notin E \setminus \{z_0\}$. Therefore there exists a number $a > 1$ such that

$$\Phi_{E \setminus \{z_0\}}(z_0) > a.$$

Consequently we can also find a polynomial $p \in \mathcal{P}$ such that $\deg p \geq 1$, $\|p\|_{E\setminus\{z_0\}} \leq 1$ and $|p(z_0)| > a^{\deg p}$. Now for $n \in \mathbb{N}$ we put $q_n(z) := \big(p(z)\big)^n \cdot (z - z_0)$ so that $q_n \in \mathcal{P}_{n\cdot\deg p+1}$. We see that

$$\|q_n\|_{E\setminus\{z_0\}} \leq \|p\|^n_{E\setminus\{z_0\}} \cdot \sup_{z\in E\setminus\{z_0\}} |z - z_0| \leq \operatorname{diam} E,$$

$$q_n(z_0) = 0,$$
$$\|q_n\|_E \leq \operatorname{diam} E,$$
$$q_n'(z) = \big(p(z)\big)^n + n \cdot p'(z) \cdot \big(p(z)\big)^{n-1} \cdot (z - z_0),$$
$$|q_n'(z_0)| = |p(z_0)|^n > a^{n\cdot\deg p}.$$

Finally, by applying GMI for the set E, we obtain for all $n \in \mathbb{N}$

$$a^{n\cdot\deg p} < |q_n'(z_0)| \leq \|q_n'\|_E \leq M \cdot (n \cdot \deg p + 1)^k \cdot \|q_n\|_E \leq M \cdot \operatorname{diam} E \cdot (n \cdot \deg p + 1)^k,$$

which is clearly impossible. $\square$

Definition 1.22 [cf. Tidten 2, definition 2]. A compact set $E \subset\subset \mathbb{C}$ is called <u>m-perfect</u>, where $m \geq 1$, if

$$\exists c \geq 1 \quad \forall z_0 \in E \quad \forall 0 < r \leq 1 \quad : \quad \left\{ z \in E \;:\; \frac{r^m}{c} \leq |z - z_0| \leq r \right\} \neq \emptyset.$$

CHAPTER II

LOCAL MARKOV INEQUALITY (LMI)

Definition 2.1 [cf. Eggink, definition 5.1; cf. Wallin-Wingren; cf. Jonsson-Wallin, chapter II §2 definition 2]. A compact set $E \subset\subset \mathbb{C}$ admits the Local Markov Inequality LMI(m,k), where $m,k \geq 1$, if

$$\forall n \in \mathbb{N} \quad \exists c_n \geq 1 \quad \forall z_0 \in E \quad \forall 0 < r \leq 1 \quad \forall p \in \mathcal{P}_n \quad : \\ \|p'\|_{E \cap B(z_0,r)} \leq \frac{c_n}{r^m} \cdot \|p\|_{E \cap B(z_0,r)}$$

and additionally $c_n \leq c_1 \cdot n^k$. Without the last assumption we speak of the Weak Local Markov Inequality WLMI(m).

Definition 2.2 [cf. Wallin-Wingren; cf. Jonsson-Wallin]. A compact set $E \subset\subset \mathbb{C}$ admits the Surround Markov Inequality SMI(m,k), where $m,k \geq 1$, if

$$\forall n \in \mathbb{N} \quad \exists c_n \geq 1 \quad \forall z_0 \in E \quad \forall 0 < r \leq 1 \quad \forall p \in \mathcal{P}_n \quad : \\ \|p'\|_{B(z_0,r)} \leq \frac{c_n}{r^m} \cdot \|p\|_{E \cap B(z_0,r)}$$

and additionally $c_n \leq c_1 \cdot n^k$. Without the last assumption we speak of the Weak Surround Markov Inequality WSMI(m).

Definition 2.3 [Eggink, definition 5.2; cf. Jonsson-Wallin, chapter II §2 proposition 2]. For any closed ball $B := B(z_0, r)$, where $z_0 \in \mathbb{C}$, $r > 0$ and $m \geq 1$ we define the following norms on $\mathcal{P}$:

$$|p|_B^m := \sum_j \frac{|p^{(j)}(z_0)|}{j!} \cdot r^{m \cdot j}.$$

Note that this is a finite sum. We also denote $|p|_B := |p|_B^1$.

Proposition 2.4 [cf. Eggink, theorem 5.3; cf. Jonsson-Wallin, chapter II §2 proposition 2]. *For a fixed compact set $E \subset\subset \mathbb{C}$, $m \geq 1$ and $n \in \mathbb{N}$, we consider the following conditions:*

(i) $$\exists c \geq 1 \quad \forall B \quad \forall p \in \mathcal{P}_n \quad : \quad |p|_B \leq \frac{c}{r^{m-1}} \cdot \|p\|_{E \cap B},$$

(ii) *E admits* WSMI(m) *for polynomials of degree n,*

(iii) *E admits* WLMI(m) *for polynomials of degree n,*

(iv) $$\exists c \geq 1 \quad \forall B \quad \forall p \in \mathcal{P}_n \quad : \quad |p|_B^m \leq c \cdot \|p\|_{E \cap B},$$

(v) $$\exists c \geq 1 \quad \forall B \quad \forall p \in \mathcal{P}_n \quad : \quad \|p'\|_{B(z_0,r^m)} \leq \frac{c \cdot n}{r^m} \cdot \|p\|_{E \cap B},$$

(vi) $$\exists \widetilde{c} \geq 1 \quad \forall B \quad \forall p \in \mathcal{P}_n \quad \forall j = 1, \ldots, n \quad : \quad |p^{(j)}(z_0)| \leq \widetilde{c} \cdot \left(\frac{n}{r^m}\right)^j \cdot \|p\|_{E \cap B},$$

(vii) *E admits* WSMI(m) *for polynomials of degree* 1,

(viii) *E is m-perfect.*

Here and further in this chapter B stands for a closed ball $B(z_0, r)$, where $z_0 \in E$ and $0 < r \leq 1$.
We assert that (i) $\Longleftrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv) $\Longleftrightarrow$ (v) $\Longleftrightarrow$ (vi) $\Longrightarrow$ (vii) $\Longleftrightarrow$ (viii).

Remark 2.5. Note that trivially we have $\|p\|_{E\cap B} \leq \|p\|_B \leq |p|_B$. Therefore in the case of a set admitting WLMI(1), the theorem implies the equivalence of all pairs of norms $(\|\cdot\|_{E\cap B}, \|\cdot\|_B, |\cdot|_B)$ of the space $\mathcal{P}_n$, uniformly with respect to B.

Also when $n = 1$ all contemplated conditions are equivalent.

Proof of proposition 2.4. (i)$\Longrightarrow$(ii) We first note that

$$|p'|_B = \sum_{0\leq j\leq n-1} \frac{|p^{(j+1)}(z_0)|}{j!} \cdot r^j = \frac{1}{r} \cdot \sum_{0\leq j\leq n-1} (j+1) \cdot \frac{|p^{(j+1)}(z_0)|}{(j+1)!} \cdot r^{j+1} \leq$$

$$\leq \frac{n}{r} \cdot \sum_{1\leq j\leq n} \frac{|p^{(j)}(z_0)|}{j!} \cdot r^j \leq \frac{n}{r} \cdot |p|_B.$$

Therefore for an arbitrary ball B and polynomial $p \in \mathcal{P}_n$, we can deduce from remark 2.5 and the assumption that

$$\|p'\|_B \leq |p'|_B \leq \frac{n}{r} \cdot |p|_B \leq \frac{c \cdot n}{r^m} \cdot \|p\|_{E\cap B}.$$

This implies WSMI(m) for polynomials of degree n, where we put $c_n := c \cdot n$.

(i)$\Longleftarrow$(ii) By the assumption we have

$$\exists c_n \geq 1 \quad \forall B \quad \forall p \in \mathcal{P}_n \ : \ \|p'\|_B \leq \frac{c_n}{r^m} \cdot \|p\|_{E\cap B}.$$

Applying Cauchy's integral formula to the polynomial p' we obtain for $j = 1, \ldots, n$

$$p^{(j)}(z_0) = \frac{(j-1)!}{2\pi i} \cdot \int_{\partial B} \frac{p'(\zeta)}{(\zeta - z_0)^j}\, d\zeta,$$

$$|p^{(j)}(z_0)| \leq \frac{(j-1)!}{2\pi} \cdot \int_{\partial B} \frac{\|p'\|_B}{|\zeta - z_0|^j} |d\zeta| = \frac{(j-1)!}{r^{j-1}} \cdot \|p'\|_B \leq \frac{(j-1)! \cdot c_n}{r^{m+j-1}} \cdot \|p\|_{E\cap B}.$$

Therefore we have $\frac{|p^{(j)}(z_0)|}{j!} \cdot r^j \leq \frac{c_n}{r^{m-1}} \cdot \|p\|_{E\cap B}$ and this is obviously also true in the case that $j = 0$. This way we obtain

$$|p|_B \leq \sum_{0\leq j\leq n} \frac{c_n}{r^{m-1}} \cdot \|p\|_{E\cap B} \leq \frac{(n+1) \cdot c_n}{r^{m-1}} \cdot \|p\|_{E\cap B}$$

and it suffices to put $c := (n+1) \cdot c_n$.

(ii)$\Longrightarrow$(iii) This follows straight from the definitions.

(iii)$\Longrightarrow$(iv) By the assumption we have

$$\exists c_n \geq 1 \quad \forall B \quad \forall p \in \mathcal{P}_n \quad : \quad \|p'\|_{E\cap B} \leq \frac{c_n}{r^m} \cdot \|p\|_{E\cap B}.$$

Iterating this inequality we obtain for $j = 0, \ldots, n$

$$\|p^{(j)}\|_{E\cap B} \leq \left(\frac{c_n}{r^m}\right)^j \cdot \|p\|_{E\cap B}$$

and consequently

$$\frac{|p^{(j)}(z_0)|}{j!} \cdot r^{m\cdot j} \leq \frac{c_n^j}{j!} \cdot \|p\|_{E\cap B},$$

$$|p|_B^m \le \sum_{0\le j\le n} \frac{c_n^j}{j!}\cdot \|p\|_{E\cap B} \le e^{c_n}\cdot \|p\|_{E\cap B}.$$

Therefore it suffices to put $c := e^{c_n}$.

(iv)$\Longrightarrow$(v) We apply GMI for the ball $B(z_0, r^m)$, remark 2.5 and the assumption to see that

$$\|p'\|_{B(z_0,r^m)} \le \frac{n}{r^m}\cdot \|p\|_{B(z_0,r^m)} \le \frac{n}{r^m}\cdot |p|_{B(z_0,r^m)} = \frac{n}{r^m}\cdot |p|_B^m \le \frac{c\cdot n}{r^m}\cdot \|p\|_{E\cap B}.$$

(v)$\Longrightarrow$(vi) Again we apply GMI for the ball $B(z_0, r^m)$ and the assumption to obtain

$$|p^{(j)}(z_0)| \le \|p^{(j)}\|_{B(z_0,r^m)} \le \left(\frac{n}{r^m}\right)^{j-1}\cdot \|p'\|_{B(z_0,r^m)} \le c\cdot \left(\frac{n}{r^m}\right)^{j}\cdot \|p\|_{E\cap B},$$

so it suffices to put $\widetilde{c} := c$.

(vi)$\Longrightarrow$(iv) We see that

$$|p|_B^m = \sum_{0\le j\le n} \frac{|p^{(j)}(z_0)|}{j!}\cdot r^{m\cdot j} \le \sum_{0\le j\le n} \frac{\widetilde{c}}{j!}\cdot \left(\frac{n}{r^m}\right)^{j}\cdot r^{m\cdot j}\cdot \|p\|_{E\cap B} \le \widetilde{c}\cdot e^n\cdot \|p\|_{E\cap B},$$

and in this case we put $c := \widetilde{c}\cdot e^n$.

(vi)$\Longrightarrow$(vii) It suffices to note that $|p'(z_0)| = \|p'\|_B$ if $p\in\mathcal{P}_1$, so we can put $c_1 := \widetilde{c}\cdot n$.

(vii)$\Longrightarrow$(viii) By applying the assumption to $p(z) := z - z_0$ and $j = 1$ we see that

$$1 \le \frac{c_1}{r^m}\cdot \sup_{z\in E\cap B} |z - z_0|,$$

which is the same as m-perfectness.

(vii)$\Longleftarrow$(viii) The assumption that the set E is m-perfect implies that

$$\exists 0 < c \le 1 \quad \forall z_0\in E \quad \forall 0 < r \le 1 \quad \exists z_1 \in E\cap B(z_0, r) \quad : \quad |z_1 - z_0| \ge c\cdot r^m.$$

Fix arbitrary $z_0\in E$, $0 < r \le 1$ and $p\in\mathcal{P}_1$, so that $p(z) = p'(z_0)\cdot(z - z_0) + p(z_0)$. Find a point $z_1 \in E\cap B$ such that $|z_1 - z_0| \ge c\cdot r^m$. Then we can assert that

$$\|p\|_{E\cap B} \ge \max\big\{|p(z_0)|, |p(z_1)|\big\} \ge \frac{1}{2}\cdot\big(|p(z_0)| + |p(z_1)|\big) \ge \frac{1}{2}\cdot |p(z_1) - p(z_0)| =$$
$$= \frac{1}{2}\cdot |p'(z_0)|\cdot |z_1 - z_0| \ge \frac{c}{2}\cdot r^m\cdot |p'(z_0)| = \frac{c}{2}\cdot r^m\cdot \|p'\|_B.$$

This is equivalent to WSMI(m) for polynomials of degree 1. $\square$

We now see that we can partly generalise proposition 2.4 to the stronger property SMI(m), where $m\ge 1$, by allowing for a variable $n\in\mathbb{N}$ while controlling the constants $c = c(n)$.

Corollary 2.6. *For a fixed compact set $E \subset\subset \mathbb{C}$ and $m, k \ge 1$, we consider the following conditions:*

(i) $\quad$ *E admits* SMI(m,k),

(ii) $$\exists c\ge 1 \quad \forall B \quad \forall n\in\mathbb{N} \quad \forall p\in\mathcal{P}_n \quad : \quad |p|_B \le \frac{c\cdot n^{k+1}}{r^{m-1}}\cdot \|p\|_{E\cap B},$$

(iii) $\quad$ *E admits* SMI$(m,k+2)$,

(iv) $$\exists \widetilde{c}\ge 1 \quad \forall B \quad \forall n\in\mathbb{N} \quad \forall p\in\mathcal{P}_n \quad : \quad |p|_B^m \le \widetilde{c}\cdot n^{k+1}\cdot \|p\|_{E\cap B}.$$

We have (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) and (ii) $\Longrightarrow$ (iv).

Proof. (i)$\Longrightarrow$(ii)$\Longrightarrow$(iii) This follows straight from the proof of (i)$\Longleftrightarrow$(ii) in proposition 2.4. Note the slight deterioration in the coefficient k.

(ii)$\Longrightarrow$(iv) This follows from the simple observation that $|p|_B^m \le r^{m-1}\cdot |p|_B + |p(z_0)|$. $\square$

Proposition 2.7 [cf. Białas-Eggink 1, proposition 4.2]. *If a compact set $E \subset\subset \mathbb{C}$ is m-perfect, where $1 \le m < 2$, then we have*

$$\forall n \in \mathbb{N} \quad \exists m_n \ge 1 \quad \exists c_n \ge 1 \quad \forall B \quad \forall p \in \mathcal{P}_n \quad : \quad |p|_B \le \frac{c_n}{r^{m_n-1}} \cdot \|p\|_{E\cap B}.$$

Proof. Proposition 2.4 implies that the set E admits WSMI(m) for polynomials of degree 1, i.e.

$$\exists c \ge 1 \quad \forall B \quad \forall p \in \mathcal{P}_1 \quad : \quad \|p'\|_B \le \frac{c}{r^m} \cdot \|p\|_{E\cap B}.$$

Furthermore for $m_1 := m \ge 1$ and $c_1 := 2c \ge 1$ we see that

$$\forall B \quad \forall p \in \mathcal{P}_1 \quad : \quad |p|_B \le \frac{c_1}{r^{m_1-1}} \cdot \|p\|_{E\cap B}.$$

In order to prove the assertion by mathematical induction, let's assume that for some $n \in \mathbb{N}$, where $n \ge 2$, we have already proved that

$$\exists m_{n-1} \ge 1 \quad \exists c_{n-1} \ge 1 \quad \forall B \quad \forall p \in \mathcal{P}_{n-1} \quad : \quad |p|_B \le \frac{c_{n-1}}{r^{m_{n-1}-1}} \cdot \|p\|_{E\cap B}.$$

We put $m_n := \frac{2\cdot m_{n-1}}{2-m} - 1 \ge 1$. Fix arbitrarily a ball B and $p \in \mathcal{P}_n$, so that we can write

$$p(z) = \sum_{0\le j\le n} a_j \cdot (z - z_0)^j.$$

If $|a_0| \ge \sum_{1\le j\le n} |a_j| \cdot r^j$ then we see that

$$|p|_B = |a_0| + \sum_{1\le j\le n} |a_j| \cdot r^j \le 2 \cdot |a_0| = 2 \cdot |p(z_0)| \le 2 \cdot \|p\|_{E\cap B} \le \frac{c_n}{r^{m_n-1}} \cdot \|p\|_{E\cap B},$$

provided that we put $c_n \ge 2$. Alternatively, in the case that $|a_0| < \sum_{1\le j\le n} |a_j| \cdot r^j$, then we have

$$|p'|_B = \sum_{1\le j\le n} j \cdot |a_j| \cdot r^{j-1} \ge \frac{1}{r} \cdot \sum_{1\le j\le n} |a_j| \cdot r^j > \frac{1}{2r} \cdot \sum_{0\le j\le n} |a_j| \cdot r^j = \frac{1}{2r} \cdot |p|_B.$$

We denote $\widetilde{B} := B\left(z_0, \frac{r}{2}\right)$. From the inductive hypothesis it follows that

$$\|p'\|_{E\cap\widetilde{B}} \ \ge \ \frac{(r/2)^{m_{n-1}-1}}{c_{n-1}} \cdot |p'|_{\widetilde{B}} \ \ge \ \frac{r^{m_{n-1}-1}}{c_{n-1}\cdot 2^{m_{n-1}-1}} \cdot \frac{1}{2^{n-1}} \cdot |p'|_B \ > \ \frac{r^{m_{n-1}-2}}{c_{n-1}\cdot 2^{m_{n-1}+n-1}} \cdot |p|_B,$$

because $p' \in \mathcal{P}_{n-1}$. We can therefore find a point $z_1 \in E \cap \widetilde{B}$ such that

$$|p'(z_1)| > \frac{r^{m_{n-1}-2}}{c_{n-1}\cdot 2^{m_{n-1}+n-1}} \cdot |p|_B = c \cdot M \cdot r^{m_{n-1}-2} \cdot |p|_B,$$

where we denote $M := \frac{1}{c\cdot c_{n-1}\cdot 2^{m_{n-1}+n-1}} < 1$. We also put $\varepsilon := \left(\frac{M\cdot m}{2e^n}\right)^{\frac{1}{2-m}} < \frac{1}{2}$ and

$$B_\varepsilon := B\left(z_1, \varepsilon \cdot r^{(m_n+1)/2}\right) \subset B.$$

Let $q(z) := (z - z_1) \cdot p'(z_1)$. By applying WSMI($m$) to $q \in \mathcal{P}_1$ we obtain

$$\frac{c}{\left(\varepsilon \cdot r^{(m_n+1)/2}\right)^m} \cdot \|q\|_{E\cap B_\varepsilon} \ge \|q'\|_{B_\varepsilon} = |p'(z_1)| > c \cdot M \cdot r^{m_{n-1}-2} \cdot |p|_B$$

and thus there exists a point $z_\varepsilon \in E \cap B_\varepsilon$ such that

$$|q(z_\varepsilon)| > M \cdot \varepsilon^m \cdot r^{m_{n-1}-2+m\cdot(m_n+1)/2} \cdot |p|_B = M \cdot \varepsilon^m \cdot r^{m_n-1} \cdot |p|_B,$$

because $m_{n-1} - 2 + m \cdot (m_n+1)/2 = \frac{2-m}{2} \cdot (m_n+1) - 2 + m \cdot (m_n+1)/2 = m_n - 1$. Note that M and ε do not depend on the choice of B and p.

We will estimate $p(z)$ by developing it into a Taylor series around the point z_1:

$$p(z) = p(z_1) + q(z) + r(z),$$

where $r(z) = \sum_{2\le j\le n} \frac{p^{(j)}(z_1)}{j!} \cdot (z-z_1)^j$. Naturally we have for $j = 1, \dots, n$

$$p^{(j)}(z) = \sum_{j\le \ell\le n} a_\ell \cdot \ell \cdot (\ell-1) \cdot \ldots \cdot (\ell-j+1) \cdot (z-z_0)^{\ell-j}$$

and therefore for arbitrary $z \in B$ we obtain

$$|p^{(j)}(z)| \le n^j \cdot \sum_{j\le\ell\le n} |a_\ell| \cdot |z-z_0|^{\ell-j} \le n^j \cdot \sum_{j\le\ell\le n} |a_\ell| \cdot r^{\ell-j} \le \left(\frac{n}{r}\right)^j \cdot \sum_{0\le\ell\le n} |a_\ell| \cdot r^\ell = \left(\frac{n}{r}\right)^j \cdot |p|_B.$$

In particular $|p^{(j)}(z_1)| \le \left(\frac{n}{r}\right)^j \cdot |p|_B$ and hence for arbitrary $z \in B_\varepsilon$ we have

$$|r(z)| \le \sum_{2\le\ell\le n} \left(\frac{n}{r}\right)^j \cdot |p|_B \cdot \frac{1}{j!} \cdot \left(\varepsilon \cdot r^{(m_n+1)/2}\right)^j =$$
$$= \sum_{2\le j\le n} \frac{n^j}{j!} \cdot \left(\varepsilon \cdot r^{(m_n-1)/2}\right)^j \cdot |p|_B \le e^n \cdot \varepsilon^2 \cdot r^{m_n-1} \cdot |p|_B.$$

We denote $C := \frac{1}{2} \cdot \left(M \cdot \varepsilon^m - e^n \cdot \varepsilon^2\right) = \frac{1}{2} \cdot \varepsilon^m \cdot (M - e^n \cdot \varepsilon^{2-m}) = \frac{1}{2} \cdot \varepsilon^m \cdot M \cdot \left(1 - \frac{m}{2}\right)$ independently of the choice of B and p and we note that $0 < C < \frac{1}{2}$. Now if $|p(z_1)| \ge C \cdot r^{m_n-1} \cdot |p|_B$ then

$$|p|_B \le \frac{1/C}{r^{m_n-1}} \cdot |p(z_1)| \le \frac{1/C}{r^{m_n-1}} \cdot \|p\|_{E\cap B}.$$

Alternatively if $|p(z_1)| < C \cdot r^{m_n-1} \cdot |p|_B$ then we see that

$$|p(z_\varepsilon)| = |p(z_1) + q(z_\varepsilon) + r(z_\varepsilon)| \ge |q(z_\varepsilon)| - |p(z_1)| - |r(z_\varepsilon)| >$$
$$> M \cdot \varepsilon^m \cdot r^{m_n-1} \cdot |p|_B - C \cdot r^{m_n-1} \cdot |p|_B - e^n \cdot \varepsilon^2 \cdot r^{m_n-1} \cdot |p|_B =$$
$$= \left(M \cdot \varepsilon^m - C - e^n \cdot \varepsilon^2\right) \cdot r^{m_n-1} \cdot |p|_B = C \cdot r^{m_n-1} \cdot |p|_B,$$

which also implies that

$$|p|_B \le \frac{1/C}{r^{m_n-1}} \cdot |p(z_\varepsilon)| \le \frac{1/C}{r^{m_n-1}} \cdot \|p\|_{E\cap B}.$$

Finally we conclude the inductive step by putting, independently of the choice of B and p,

$$c_n := \frac{1}{C} > 2. \quad \square$$

Remark 2.8. A careful inspection of the constants reveals that

$$m_n = \frac{m^2+m-2}{m} \cdot \left(\frac{2}{2-m}\right)^{n-1} + \frac{2-m}{m},$$

$$c_n = \frac{4 \cdot 2^{m_n}}{2-m} \cdot \left(\frac{c^2 \cdot c_{n-1}^2 \cdot (4e^m)^n}{m^m}\right)^{\frac{1}{2-m}}.$$

By denoting $\mu := \frac{2}{2-m} \ge 2$ we obtain an estimate for c_n:

$$c_n \le 2\mu \cdot 2^{m_n} \cdot \left(c \cdot c_{n-1} \cdot (2e)^n\right)^\mu.$$

However, if $m = 1$ then for all $n \in \mathbb{N}$ we have $m_n = 1$ and $c_n = 8 \cdot c^2 \cdot c_{n-1}^2 \cdot (4e)^n$.

By combining the results of propositions 2.4 and 2.7 we obtain:

Corollary 2.9 [Białas-Eggink 1, corollary 4.4; cf. Wallin-Wingren]. *If $E \subset\subset \mathbb{C}$ is an m-perfect set and $1 \leq m < 2$, then for all $n \in \mathbb{N}$ there exists $m_n \geq 1$ such that the set E admits* WSMI(m_n) *for polynomials of degree n.*

Specifically in the case of $m = 1$ we have:

Corollary 2.10 [cf. Jonsson-Wallin, chapter II §2 proposition 4]. *A compact set $E \subset\subset \mathbb{C}$ admits* WSMI(1) *if and only if it is uniformly perfect, i.e.* 1*-perfect.*

CHAPTER III

LOCAL MARKOV PROPERTY (LMP)

Whereas many different versions of local Markov-type polynomial inequalities are covered in the literature, the version introduced by L.P. Bos and P.D. Milman seems to be the most universal.

DEFINITION 3.1 [cf. Białas-Eggink 1, definition 1.2; cf. Bos-Milman, definition 2.3]. A compact set $E \subset\subset \mathbb{C}$ admits the Local Markov Property $\mathrm{LMP}(m,k)$, where $m,k \geq 1$, if

$$\forall n \in \mathbb{N} \quad \exists c_n \geq 1 \quad \forall z_0 \in E \quad \forall 0 < r \leq 1 \quad \forall p \in \mathcal{P}_n \quad \forall j = 1,\dots,n \quad : \\ |p^{(j)}(z_0)| \leq \left(\frac{c_n}{r^m}\right)^j \cdot \|p\|_{E \cap B(z_0,r)}$$

and additionally $c_n \leq c_1 \cdot n^k$. Without the last assumption we speak of the Weak Local Markov Property $\mathrm{WLMP}(m)$. We will write that the set E admits LMP, respectively WLMP, if it admits $\mathrm{LMP}(m,k)$, respectively $\mathrm{WLMP}(m)$, for some $m,k \geq 1$.

REMARK 3.2. L.P. Bos and P.D. Milman use a longer construction $\exists r_0 > 0 \ \dots \ \forall 0 < r \leq r_0 \dots$. This is clearly equivalent to our definition.

We also see that this property is invariant to a linear change of the variable. This in turn implies that we can split up a compact set $E = A \cup B$, such that $A \cap B = \emptyset$, in the sense that if the set E admits LMP or WLMP, then both sets A and B admit the same property. Obviously the converse is true too.

Furthermore the following proposition proves that in the definition of the Local Markov Property we can restrict ourselves to $j = 1$, albeit with a deterioration of the constants c_n.

PROPOSITION 3.3 [Białas-Eggink 2, proposition 2.6]. *If for a compact set $E \subset\subset \mathbb{C}$ and $m \geq 1$ we have*

$$\forall n \in \mathbb{N} \quad \exists c_n \geq 1 \quad \forall z_0 \in E \quad \forall 0 < r \leq 1 \quad \forall p \in \mathcal{P}_n \quad : \\ |p'(z_0)| \leq \frac{c_n}{r^m} \cdot \|p\|_{E \cap B(z_0,r)}$$

then the set E admits $\mathrm{WLMP}(m)$. If additionally $c_n \leq c_1 \cdot n^k$, where $k \geq 1$, then the set E admits $\mathrm{LMP}(m, k+m)$.

PROOF. Fix arbitrary $n \in \mathbb{N}$, $z_0 \in E$, $0 < r \leq 1$, $p \in \mathcal{P}_n$ and $j \in \{2,\dots,n\}$. By applying the assumption to the derivative $p^{(j-1)} \in \mathcal{P}_n$ and radius r/n we see that

$$|p^{(j)}(z_0)| \leq \frac{c_n \cdot n^m}{r^m} \cdot \|p^{(j-1)}\|_{E \cap B(z_0,r/n)}.$$

Let z_1 be a point of $E \cap B(z_0, r/n)$ such that $|p^{(j-1)}(z_1)| = \|p^{(j-1)}\|_{E \cap B(z_0,r/n)}$. Next we obtain similarly

$$|p^{(j)}(z_0)| \leq \frac{c_n \cdot n^m}{r^m} \cdot |p^{(j-1)}(z_1)| \leq \left(\frac{c_n \cdot n^m}{r^m}\right)^2 \cdot \|p^{(j-2)}\|_{E \cap B(z_1,r/n)}.$$

We continue in this fashion to obtain points $z_1, z_2, \dots, z_{j-1}$ such that for all $\ell = 1,\dots,j-1$ we have $z_\ell \in E \cap B(z_{\ell-1}, r/n)$ and $|p^{(j-\ell)}(z_\ell)| = \|p^{(j-\ell)}\|_{E \cap B(z_{\ell-1},r/n)}$. This way we conclude that

$$|p^{(j)}(z_0)| \leq \left(\frac{c_n \cdot n^m}{r^m}\right)^j \cdot \|p\|_{E \cap B(z_{j-1},r/n)} \leq \\ \leq \left(\frac{c_n \cdot n^m}{r^m}\right)^j \cdot \|p\|_{E \cap B(z_0, j \cdot r/n)} \leq \left(\frac{c_n \cdot n^m}{r^m}\right)^j \cdot \|p\|_{E \cap B(z_0,r)},$$

which proves WLMP(m) or LMP($m, k+m$), depending on the assumption regarding c_n. $\square$

REMARK 3.4. If a compact set $E \subset\subset \mathbb{C}$ admits WLMP(m), where $m \geq 1$, then for all $z_0 \in E$, $0 < r \leq 1$, $n \in \mathbb{N}$ and $p \in \mathcal{P}_n$ we have

$$|p|^m_{B(z_0,r)} = \sum_{0\leq j\leq n} \frac{|p^{(j)}(z_0)|}{j!} \cdot r^{m\cdot j} \leq \sum_{0\leq j\leq n} \frac{1}{j!} \cdot \left(\frac{c_n}{r^m}\right)^j \cdot \|p\|_{E\cap B(z_0,r)} \cdot r^{m\cdot j} \leq$$

$$\leq \sum_{0\leq j\leq n} \frac{1}{j!} \cdot c_n^n \cdot \|p\|_{E\cap B(z_0,r)} \leq e \cdot c_n^n \cdot \|p\|_{E\cap B(z_0,r)}.$$

Compare this with proposition 2.4.

REMARK 3.5. Obviously for any compact set $E \subset\subset \mathbb{C}$ and $m, k \geq 1$ we have the following strings of implications:

$$\begin{array}{ccccccc}
\mathrm{SMI}(m,k) & \Longrightarrow & \mathrm{LMI}(m,k) & \Longrightarrow & \mathrm{LMP}(m,k) & \Longrightarrow & \mathrm{GMI}(k), \\
\Downarrow & & \Downarrow & & \Downarrow & & \\
\mathrm{WSMI}(m) & \Longrightarrow & \mathrm{WLMI}(m) & \Longrightarrow & \mathrm{WLMP}(m) & \Longrightarrow & m-\text{perfectness}.
\end{array}$$

The vertical implications as well as the first horizontal implications follow straight from the definitions. The second horizontal implication is obtained by iterating LMI(m, k), respectively WLMI(m), j times in order to estimate $\|p^{(j)}\|_{E\cap B(z_0,r)}$, like in the proof of proposition 2.4. This way, unlike in proposition 3.3, the estimate for c_n does not deteriorate. The upper third horizontal implication is obtained by putting $r = 1$ and $j = 1$. The lower third horizontal implication follows from applying WLMP(m) with $j = 1$ to the polynomial $p(z) = z - z_0$.

CHAPTER IV

POMMERENKE PROPERTY (PP)

Definition 4.1 [cf. Pommerenke, theorem 1]. A compact set $E \subset\subset \mathbb{C}$ admits the Pommerenke Property PP(m), where $m \geq 1$, if

$$\exists 0 < c \leq 1 \quad \forall z_0 \in E \quad \forall 0 < r \leq 1 \quad : \quad \operatorname{cap}\big(E \cap B(z_0, r)\big) \geq c \cdot r^m.$$

Remark 4.2. If a compact set $E \subset\subset \mathbb{C}$ admits PP(m), where $m \geq 1$, then obviously it must be m-perfect. To see this we use the fact that the logarithmic capacity of a ball is equal to its radius, which can easily be deduced from theorem 1.11. Hence for arbitrary $z_0 \in E$ and $0 < r \leq 1$ we have

$$\operatorname{cap}\big(E \cap B(z_0, r)\big) \geq c \cdot r^m > \frac{c}{2} \cdot r^m = \operatorname{cap} B\big(z_0, \frac{c}{2} \cdot r^m\big) \geq \operatorname{cap}\Big(E \cap B\big(z_0, \frac{c}{2} \cdot r^m\big)\Big)$$

and therefore

$$\Big\{z \in E \ : \ \frac{c}{2} \cdot r^m \leq |z - z_0| \leq r\Big\} \neq \emptyset.$$

Ch. Pommerenke proved that any compact set $E \subset\subset \mathbb{C}$ admits PP(1) if and only if it is uniformly perfect. In this chapter we will introduce the much larger class of (m, s, κ)-perfect sets, where $m, s \geq 1$ and $\kappa \in \mathbb{N} \backslash \{1\}$. Subsequently we will use these sets to prove that any compact set admitting WLMP(m), with some $m \geq 1$, also admits PP(m') for any $m' > m^2$. In particular, by the Wiener criterion, this implies L-regularity.

Definition 4.3 [cf. Białas-Eggink 1, section 3; cf. Eggink, definition 4.1]. Denote by B a closed ball with diameter $0 < \operatorname{diam} B \leq 1$. We take smaller balls $B_{i_1} \subset B$, where $i_1 = 1, 2, \ldots, \kappa$ for some $\kappa \in \mathbb{N} \setminus \{1\}$. Subsequently we take even smaller balls $B_{i_1, i_2} \subset B_{i_1}$, where $i_1, i_2 = 1, 2, \ldots, \kappa$ and so on. We put

$$E^\ell := \bigcup_{i_1, \ldots, i_\ell}^{1, \ldots, \kappa} B_{i_1, \ldots, i_\ell}, \qquad E := \bigcap_{\ell=1}^{\infty} E^\ell.$$

If there exist $m, s \geq 1$, $0 < a \leq 1$ and $0 < b \leq 1$ such that for all $\ell \in \mathbb{Z}_+$ and $i_1, \ldots, i_{\ell+1} = 1, 2, \ldots, \kappa$ we have

$$\operatorname{diam} B_{i_1, \ldots, i_{\ell+1}} = a \cdot (\operatorname{diam} B_{i_1, \ldots, i_\ell})^m, \tag{1}$$

$$\begin{gathered}\operatorname{dist}(B_{i_1, \ldots, i_\ell, \mu}, B_{i_1, \ldots, i_\ell, \nu}) \geq b \cdot (\operatorname{diam} B_{i_1, \ldots, i_\ell})^s \\ \text{for all } \mu, \nu = 1, \ldots, \kappa \text{ such that } \mu \neq \nu,\end{gathered} \tag{2}$$

then we will call the set E an elementary (m, s, κ)-perfect set. Here and in the sequel for $\ell = 0$ the symbols $B_{i_1, \ldots, i_\ell}$ and $B_{i_1, \ldots, i_\ell, \mu}$ stand for the sets B and B_μ, respectively.

Such a set does not have to exist for arbitrary constants m, s, κ, a and b. In order for κ smaller balls to fit in a bigger ball, it is definitely sufficient to require that $a, b \leq \frac{1}{2\kappa}$.

Elementary $(1, 1, \kappa)$-perfect sets were studied by M. Tsuji under the name 'general planar Cantor set' in [Tsuji 1; see also Tsuji 2, chapter III §16.2], where he gave an estimate for their logarithmic capacity and proved that they are L-regular. It turned out that his proof can easily be generalized for elementary (m, s, κ)-perfect sets, however under the condition, that $\kappa > m$.

Lemma 4.4 [cf. Białas-Eggink 1, lemma 3.5; cf. Eggink, theorem 4.2]. *Let $E \subset\subset \mathbb{C}$ be an elementary (m, s, κ)-perfect set as above, where $m, s \geq 1$ and $\kappa \in \mathbb{N}$ such that $\kappa > m$. Then we can assert that*

$$\operatorname{cap} E \geq a^{\frac{s}{\kappa-m}} \cdot b \cdot (\operatorname{diam} B)^{\frac{s\cdot(\kappa-1)}{\kappa-m}},$$

$$\forall z_0 \in E \quad \forall 0 < r \leq \operatorname{diam} B \quad : \quad \operatorname{cap}\bigl(E \cap B(z_0, r)\bigr) \geq a^{\frac{s\cdot\kappa}{\kappa-m}} \cdot b \cdot r^{\frac{m\cdot s\cdot(\kappa-1)}{\kappa-m}},$$

$$\limsup_{r\to 0} \frac{\log 1/r}{\log 1/\operatorname{cap}\bigl(E \cap B(z_0, r)\bigr)} \geq \frac{\kappa - m}{s\cdot(\kappa-1)}.$$

Proof. From equality (1) we deduce that for each $\ell \in \mathbb{Z}_+$ we have

(3) $$\begin{aligned}\operatorname{diam} B_{i_1,\dots,i_\ell} &= a \cdot (\operatorname{diam} B_{i_1,\dots,i_{\ell-1}})^m = a \cdot \bigl(a \cdot (\operatorname{diam} B_{i_1,\dots,i_{\ell-2}})^m\bigr)^m = \\ &= a^{1+m} \cdot (\operatorname{diam} B_{i_1,\dots,i_{\ell-2}})^{m^2} = \ldots = a^{S(\ell)} \cdot (\operatorname{diam} B)^{m^\ell},\end{aligned}$$

where $S(\ell) := 1 + m + \ldots + m^{\ell-1}$, $S(0) := 0$. From this it follows that

(4) $$\operatorname{dist}(B_{i_1,\dots,i_\ell,\mu}, B_{i_1,\dots,i_\ell,\nu}) \geq a^{s\cdot S(\ell)} \cdot b \cdot (\operatorname{diam} B)^{s\cdot m^\ell} \quad \text{for all } \mu \neq \nu.$$

The papers by M. Fekete and F. Leja concerning the transfinite diameter [Tsuji 2, chapter III §5; Leja 2, chapter 11; Fekete], teach us that for every compact set $K \subset\subset \mathbb{C}$ there exist sets of extremal points $\{z_\mu^{(N)}\}_{\mu=1,\dots,N} \subset K$, such that

$$d_N(K) := \left(\prod_{1\leq\mu<\nu\leq N} |z_\mu^{(N)} - z_\nu^{(N)}|\right)^{1/\binom{N}{2}} \xrightarrow[N\to\infty]{} \operatorname{cap} K.$$

Therefore for a fixed set $E^\ell = \bigcup_{i_1,\dots,i_\ell}^{1,\dots,\kappa} B_{i_1,\dots,i_\ell}$ we can find sets of extremal points $\{z^\mu_{i_1,\dots,i_\ell}\}_{\mu=1,\dots,N}$, dependent also on $N \in \mathbb{N} \setminus \{1\}$, located on each $B_{i_1,\dots,i_\ell}$ such that

(5) $$\left(\prod_{1\leq\mu<\nu\leq N} |z^\mu_{i_1,\dots,i_\ell} - z^\nu_{i_1,\dots,i_\ell}|\right)^{1/\binom{N}{2}} \xrightarrow[N\to\infty]{} \operatorname{cap} B_{i_1,\dots,i_\ell}.$$

This way we obtain $\kappa^\ell \cdot N$ points $\{z^\mu_{i_1,\dots,i_\ell} \ : \ i_1, \dots, i_\ell = 1, \dots, \kappa, \ \mu = 1, \dots, N\} \subset E^\ell$, for which we have (in the notation of remark 1.6)

(6) $$\bigl(d_{\kappa^\ell\cdot N}(E^\ell)\bigr)^{\binom{\kappa^\ell\cdot N}{2}} \geq \Pi_0 \cdot \Pi_1 \cdot \ldots \cdot \Pi_{\ell-1} \cdot \Pi_\ell =: \Pi,$$

where

$$\Pi_\ell := \prod_{i_1,\dots,i_\ell}^{1,\dots,\kappa} \prod_{1\leq\mu<\nu\leq N} |z^\mu_{i_1,\dots,i_\ell} - z^\nu_{i_1,\dots,i_\ell}|,$$

$$\Pi_{\ell-1} := \prod_{i_1,\dots,i_{\ell-1}}^{1,\dots,\kappa} \prod_{1\leq i_\ell<j_\ell\leq\kappa} \prod_{\mu,\nu}^{1,\dots,N} |z^\mu_{i_1,\dots,i_{\ell-1},i_\ell} - z^\nu_{i_1,\dots,i_{\ell-1},j_\ell}|,$$

$$\Pi_{\ell-2} := \prod_{i_1,\dots,i_{\ell-2}}^{1,\dots,\kappa} \prod_{1\leq i_{\ell-1}<j_{\ell-1}\leq\kappa} \prod_{i_\ell,j_\ell}^{1,\dots,\kappa} \prod_{\mu,\nu}^{1,\dots,N} |z^\mu_{i_1,\dots,i_{\ell-2},i_{\ell-1},i_\ell} - z^\nu_{i_1,\dots,i_{\ell-2},j_{\ell-1},j_\ell}|,$$

$$\cdots$$

$$\cdots$$

$$\cdots$$

$$\Pi_0 := \prod_{1\le i_1<j_1\le\kappa} \prod_{\substack{i_2,\dots,i_\ell \\ j_2,\dots,j_\ell}}^{1,\dots,\kappa} \prod_{\mu,\nu}^{1,\dots,N} |z^\mu_{i_1,i_2,\dots,i_\ell} - z^\nu_{j_1,j_2,\dots,j_\ell}|.$$

We see that $\Pi = \Pi_\ell \cdot \Pi_{\ell-1} \cdot \ldots \cdot \Pi_1 \cdot \Pi_0$ is the product of $\ell+1$ elements, all dependent on $N \in \mathbb{N}\setminus\{1\}$, where Π_ℓ is determined by pairs of points belonging to the same $B_{i_1,\dots,i_\ell}$, $\Pi_{\ell-1}$ is determined by pairs of points belonging to the same $B_{i_1,\dots,i_{\ell-1}}$, but to different $B_{i_1,\dots,i_{\ell-1},i_\ell}$ and $B_{i_1,\dots,i_{\ell-1},j_\ell}$, and so forth. Finally Π_0 is determined by pairs of points belonging to different B_{i_1} and B_{j_1}.

Inequality (5), the fact that the logarithmic capacity of a ball is equal to its radius and equality (3) imply that

$$\lim_{N\to\infty} \Pi_\ell^{1/\binom{N}{2}} = \prod_{i_1,\dots,i_\ell}^{1,\dots,\kappa} \operatorname{cap} B_{i_1,\dots,i_\ell} = \prod_{i_1,\dots,i_\ell}^{1,\dots,\kappa} \frac{\operatorname{diam} B_{i_1,\dots,i_\ell}}{2} = \left(\frac{a^{S(\ell)}\cdot(\operatorname{diam} B)^{m^\ell}}{2}\right)^{\kappa^\ell}. \tag{7}$$

In the case of the product Π_j, where $j = 0,\dots,\ell-1$, we use inequality (4) to obtain

$$|z^\mu_{i_1,\dots,i_j,i_{j+1},\dots,i_\ell} - z^\nu_{i_1,\dots,i_j,j_{j+1},\dots,j_\ell}| \ \ge\ \operatorname{dist}(B_{i_1,\dots,i_j,i_{j+1}}, B_{i_1,\dots,i_j,j_{j+1}}) \ \ge a^{s\cdot S(j)}\cdot b\cdot(\operatorname{diam} B)^{s\cdot m^j}.$$

The number of such pairs of points in the product Π_j is equal to $\kappa^j\cdot\binom{\kappa}{2}\cdot\kappa^{2\cdot(\ell-j-1)}\cdot N^2 = \frac{\kappa-1}{2}\cdot\kappa^{2\ell-j-1}\cdot N^2$. Consequently we have

$$\Pi_j \ge \left(a^{s\cdot S(j)}\cdot b\cdot(\operatorname{diam} B)^{s\cdot m^j}\right)^{\frac{\kappa-1}{2}\cdot\kappa^{2\ell-j-1}\cdot N^2},$$

$$\log \Pi_j \ge \frac{\kappa-1}{2\kappa}\cdot\kappa^{2\ell}\cdot N^2\cdot\frac{1}{\kappa^j}\cdot\Big(s\cdot S(j)\cdot\log a + \log b + s\cdot m^j\cdot\log\operatorname{diam} B\Big).$$

From this it follows that

$$\log(\Pi_{\ell-1}\cdot\Pi_{\ell-2}\cdot\ldots\cdot\Pi_0) = \sum_{j=0}^{\ell-1}\log\Pi_j \ge$$

$$\ge \frac{\kappa-1}{2\kappa}\cdot\kappa^{2\ell}\cdot N^2\cdot\sum_{j=0}^{\ell-1}\left(s\cdot\log a\cdot\frac{S(j)}{\kappa^j} + \frac{\log b}{\kappa^j} + s\cdot\log\operatorname{diam} B\cdot\left(\frac{m}{\kappa}\right)^j\right).$$

By putting this into inequality (6) and applying equality (7) we obtain

$$\log\ d_{\kappa^\ell\cdot N}(E^\ell) \ge \frac{2}{\kappa^\ell\cdot N\cdot(\kappa^\ell\cdot N-1)}\cdot\big(\log\Pi_\ell + \log(\Pi_{\ell-1}\cdot\ldots\cdot\Pi_0)\big),$$

$$\log\operatorname{cap} E^\ell = \lim_{N\to\infty}\log d_{\kappa^\ell\cdot N}(E^\ell) \ge \frac{1}{\kappa^{2\ell}}\cdot\log\left(\frac{a^{S(\ell)}\cdot(\operatorname{diam} B)^{m^\ell}}{2}\right)^{\kappa^\ell} +$$

$$+\frac{\kappa-1}{\kappa}\cdot\left(s\cdot\log a\cdot\sum_{j=0}^{\ell-1}\frac{S(j)}{\kappa^j} + \sum_{j=0}^{\ell-1}\frac{\log b}{\kappa^j} + s\cdot\log\operatorname{diam} B\cdot\sum_{j=0}^{\ell-1}\left(\frac{m}{\kappa}\right)^j\right).$$

Since $\kappa > m$ and $E^1 \supset E^2 \supset \ldots \supset \bigcap_{\ell=1}^\infty E^\ell = E$, we see that

$$\log\operatorname{cap} E = \lim_{\ell\to\infty}\log\operatorname{cap} E^\ell \ge$$

$$\ge \lim_{\ell\to\infty}\frac{S(\ell)\cdot\log a + m^\ell\cdot\log\operatorname{diam} B - \log 2}{\kappa^\ell} + \frac{\kappa-1}{\kappa}\cdot\left(s\cdot\log a\cdot\sum_{j=0}^{\infty}\frac{S(j)}{\kappa^j} + \frac{\log b}{1-\frac{1}{\kappa}} + \frac{s\cdot\log\operatorname{diam} B}{1-\frac{m}{\kappa}}\right).$$

It is easy to verify that

$$S(j) = \begin{cases} j & \text{if } m = 1 \\ \frac{m^j-1}{m-1} & \text{if } m > 1 \end{cases}$$

and in either case

$$\sum_{j=0}^{\infty}\frac{S(j)}{\kappa^j}=\frac{\kappa}{(\kappa-m)\cdot(\kappa-1)}.$$

Finally we obtain

$$\log\operatorname{cap}E\geq\frac{s}{\kappa-m}\cdot\log a+\log b+\frac{s\cdot(\kappa-1)}{\kappa-m}\cdot\log\operatorname{diam}B,$$
$$\operatorname{cap}E\geq a^{\frac{s}{\kappa-m}}\cdot b\cdot(\operatorname{diam}B)^{\frac{s\cdot(\kappa-1)}{\kappa-m}}.$$

Now fix an arbitrary point $z_0\in E=\bigcap_{\ell=1}^{\infty}\bigcup_{i_1,\dots,i_\ell}^{1,\dots,\kappa}B_{i_1,\dots,i_\ell}$ and $0<r\leq\operatorname{diam}B$. Find $\ell_0\in\mathbb{Z}_+$ and $i_1,\dots,i_{\ell_0}$ such that $z_0\in B_{i_1,\dots,i_{\ell_0}}$ and $a\cdot r^m<r_0:=\operatorname{diam}B_{i_1,\dots,i_{\ell_0}}\leq r$. Then $E\cap B_{i_1,\dots,i_{\ell_0}}$ is also an elementary (m,s,κ)-perfect set with the same constants $a,b>0$ as the set E and therefore we have

$$\operatorname{cap}(E\cap B_{i_1,\dots,i_{\ell_0}})\geq a^{\frac{s}{\kappa-m}}\cdot b\cdot(\operatorname{diam}B_{i_1,\dots,i_{\ell_0}})^{\frac{s\cdot(\kappa-1)}{\kappa-m}}=a^{\frac{s}{\kappa-m}}\cdot b\cdot r_0^{\frac{s\cdot(\kappa-1)}{\kappa-m}}.$$

Simultaneously we have $E\cap B(z_0,r)\supset E\cap B(z_0,r_0)\supset E\cap B_{i_1,\dots,i_{\ell_0}}$ and this leads to

$$\begin{aligned}\log\operatorname{cap}\big(E\cap B(z_0,r)\big)\geq\log\operatorname{cap}\big(E\cap B(z_0,r_0)\big)\geq\log\operatorname{cap}\big(E\cap B_{i_1,\dots,i_{\ell_0}}\big)\geq\\ \geq\log\left(a^{\frac{s}{\kappa-m}}\cdot b\right)+\frac{s\cdot(\kappa-1)}{\kappa-m}\cdot\log r_0>\log\left(a^{\frac{s}{\kappa-m}}\cdot b\right)+\frac{s\cdot(\kappa-1)}{\kappa-m}\cdot\log\left(a\cdot r^m\right)=\\ =\log\left(a^{\frac{s\cdot\kappa}{\kappa-m}}\cdot b\right)+\frac{m\cdot s\cdot(\kappa-1)}{\kappa-m}\cdot\log r.\end{aligned}$$

We conclude that

$$\operatorname{cap}\big(E\cap B(z_0,r)\big)\geq a^{\frac{s\cdot\kappa}{\kappa-m}}\cdot b\cdot r^{\frac{m\cdot s\cdot(\kappa-1)}{\kappa-m}},$$
$$\liminf_{r\to0}\frac{\log 1/r}{\log 1/\operatorname{cap}\big(E\cap B(z_0,r)\big)}\geq\frac{\kappa-m}{m\cdot s\cdot(\kappa-1)},$$

but also

$$\begin{aligned}\limsup_{r\to0}\frac{\log 1/r}{\log 1/\operatorname{cap}\big(E\cap B(z_0,r)\big)}\geq\\ \geq\lim_{r_0\to0}\frac{\log 1/r_0}{\frac{s\cdot(\kappa-1)}{\kappa-m}\cdot\log 1/r_0-\log\left(a^{\frac{s}{\kappa-m}}\cdot b\right)}=\frac{\kappa-m}{s\cdot(\kappa-1)}.\quad\square\end{aligned}$$

Definition 4.5 [cf. Białas-Eggink 1, definition 3.1; cf. Eggink, definition 4.3]. A compact set $E\subset\subset\mathbb{C}$ is called $\underline{(m,s,\kappa)-\text{perfect}}$, where $m,s\geq1$ and $\kappa\in\mathbb{N}\setminus\{1\}$, if there exist constants $0<a\leq1$ and $0<b\leq1$ such that for all $z_0\in E$ there exists an elementary (m,s,κ)-perfect set E_{z_0} with constants a,b and $\operatorname{diam}B=1$, so that $z_0\in E_{z_0}\subset E$.

If a set is (m,m,κ)-perfect, then we will call it simply $\underline{(m,\kappa)\text{-perfect}}$. Finally, if a set is (m,κ)-perfect for all $\kappa\in\mathbb{N}\setminus\{1\}$, then we will call it $\underline{(m,\infty)\text{-perfect}}$.

Theorem 4.6 [cf. Białas-Eggink 1, proposition 3.6; cf. Eggink, corollary 4.4]. *If a compact set $E\subset\subset\mathbb{C}$ is (m,s,κ)-perfect as above, where $m,s\geq1$ and $\kappa\in\mathbb{N}$ such that $\kappa>m$, then we can assert that*

$$\operatorname{cap}E\geq\operatorname{cap}E_{z_0}\geq a^{\frac{s}{\kappa-m}}\cdot b,$$
$$\forall z_0\in E\quad\forall 0<r\leq1\quad:\quad\operatorname{cap}\big(E\cap B(z_0,r)\big)\geq\operatorname{cap}\big(E_{z_0}\cap B(z_0,r)\big)\geq a^{\frac{s\cdot\kappa}{\kappa-m}}\cdot b\cdot r^{\frac{m\cdot s\cdot(\kappa-1)}{\kappa-m}},$$
$$\limsup_{r\to0}\frac{\log 1/r}{\log 1/\operatorname{cap}\big(E\cap B(z_0,r)\big)}\geq\frac{\kappa-m}{s\cdot(\kappa-1)}.$$

Consequently the set E admits PP $\left(\frac{m\cdot s\cdot(\kappa-1)}{\kappa-m}\right)$ *and by Wiener's criterion it is L-regular.*

Proof. This follows straight from the definitions, lemma 4.4 and Wiener's criterion [Tsuji 2, theorem III 62, corollary 2] . $\square$

Proposition 4.7 [cf. Białas-Eggink 1, lemmas 3.2 and 3.3; cf. Eggink, theorem 4.5]. *For a fixed compact set $E \subset\subset \mathbb{C}$, $m, s \geq 1$ and $\kappa \in \mathbb{N} \setminus \{1\}$ we consider the following conditions:*

(i) $$\exists 0 < c \leq 1 \quad \forall z_0 \in E \quad \forall 0 < r \leq 1 \quad \exists z_1, \dots, z_{\kappa-1} \in E \cap B(z_0, r) \quad : \\ |z_\mu - z_\nu| \geq c \cdot r^m \quad \textit{for all } \mu, \nu = 0, \dots, \kappa - 1 \textit{ such that } \mu \neq \nu,$$

(ii) $$E \textit{ is } (m, \kappa)\textit{-perfect},$$

(iii) $$E \textit{ is } (m, s, \kappa)\textit{-perfect},$$

(iv) $$\exists 0 < c \leq 1 \quad \forall z_0 \in E \quad \forall 0 < r \leq 1 \quad \exists z_1, \dots, z_{\kappa-1} \in E \cap B(z_0, r) \quad : \\ |z_\mu - z_\nu| \geq c \cdot r^{m \cdot s} \quad \textit{for all } \mu, \nu = 0, \dots, \kappa - 1 \textit{ such that } \mu \neq \nu.$$

We assert that (i) $\Longrightarrow$ (ii), (ii) $\Longrightarrow$ (iii) provided that $s \geq m$, and (iii) $\Longrightarrow$ (iv) regardless of $s \geq 1$.

Proof. (i) $\Longrightarrow$ (ii) For an arbitrary point $z_0 \in E$ we will construct by induction an elementary (m, m, κ)-perfect set E_{z_0} with constants $a = b = \frac{c}{2 \cdot 4^m}$ and $\operatorname{diam} B = 1$ such that $z_0 \in E_{z_0} \subset E$. We start by putting $B := B(z_0, r_0)$, where $r_0 := \frac{1}{2}$, so that $\operatorname{diam} B = 2r_0 = 1$.

Assume that we have already constructed balls $B_{i_1,\dots,i_\ell} = B(z_{i_1,\dots,i_\ell}, r_\ell)$ for all $\ell \leq \ell_0$ and $i_1, \dots, i_\ell = 1, \dots, \kappa$, as stipulated by definition 4.3, and assume that $z_{i_1,\dots,i_\ell} \in E$. Note that according to equality (1), the diameter of the ball $B_{i_1,\dots,i_\ell}$ depends only on ℓ and not on the choice of $i_1, \dots, i_\ell$.

Now for fixed $i_1, \dots, i_{\ell_0} = 1, \dots, \kappa$ we apply assumption (i) to the point $z_{i_1,\dots,i_{\ell_0},\kappa} := z_{i_1,\dots,i_{\ell_0}}$ and radius $\frac{1}{2} r_{\ell_0}$. Therefore there exist $\kappa - 1$ points, which we denote $z_{i_1,\dots,i_{\ell_0},j}$, where $j = 1, \dots, \kappa - 1$, such that

$$z_{i_1,\dots,i_{\ell_0},j} \in E \cap B\left(z_{i_1,\dots,i_{\ell_0}}, \frac{r_{\ell_0}}{2}\right) \quad \text{for } j = 1, \dots, \kappa,$$

$$|z_{i_1,\dots,i_{\ell_0},\mu} - z_{i_1,\dots,i_{\ell_0},\nu}| \geq c \cdot \left(\frac{r_{\ell_0}}{2}\right)^m \quad \text{for } \mu, \nu = 1, \dots, \kappa, \text{ such that } \mu \neq \nu.$$

We put $r_{\ell_0+1} := \frac{c}{4 \cdot 2^m} \cdot r_{\ell_0}^m = \frac{1}{2} a \cdot (2r_{\ell_0})^m$ and $B_{i_1,\dots,i_{\ell_0},j} := B(z_{i_1,\dots,i_{\ell_0},j}, r_{\ell_0+1})$ for $j = 1, \dots, \kappa$ because this way we have $\operatorname{diam} B_{i_1,\dots,i_{\ell_0},j} = 2r_{\ell_0+1} = a \cdot (2r_{\ell_0})^m = a \cdot (\operatorname{diam} B_{i_1,\dots,i_{\ell_0}})^m$ and

$$\operatorname{dist}(B_{i_1,\dots,i_{\ell_0},\mu}, B_{i_1,\dots,i_{\ell_0},\nu}) \geq |z_{i_1,\dots,i_{\ell_0},\mu} - z_{i_1,\dots,i_{\ell_0},\nu}| - 2r_{\ell_0+1} \geq \frac{c}{2 \cdot 2^m} \cdot r_{\ell_0}^m = b \cdot (\operatorname{diam} B_{i_1,\dots,i_{\ell_0}})^m$$

for all $\mu, \nu = 1, \dots, \kappa$ such that $\mu \neq \nu$. We also see that $B_{i_1,\dots,i_{\ell_0},j} \subset B_{i_1,\dots,i_{\ell_0}}$ because

$$r_{\ell_0+1} + |z_{i_1,\dots,i_{\ell_0},j} - z_{i_1,\dots,i_{\ell_0}}| \leq \frac{c}{4 \cdot 2^m} \cdot r_{\ell_0}^m + \frac{1}{2} r_{\ell_0} < r_{\ell_0}.$$

We repeat this construction for all $i_1, \dots, i_{\ell_0} = 1, \dots, \kappa$ and then we increase ℓ_0. This way we obtain an elementary (m, m, κ)-perfect set with constants a, b and $\operatorname{diam} B = 1$

$$E_{z_0} := \bigcap_{\ell=1}^{\infty} \bigcup_{i_1,\dots,i_\ell}^{1,\dots,\kappa} B_{i_1,\dots,i_\ell}.$$

Note that $z_0 \in E_{z_0}$ because $z_0 = z_\kappa = z_{\kappa,\kappa} = z_{\kappa,\kappa,\kappa} = \dots$ and therefore this point is an element of all unions $\bigcup_{i_1,\dots,i_\ell}^{1,\dots,\kappa} B_{i_1,\dots,i_\ell}$.

It remains to verify that $E_{z_0} \subset E$. For an arbitrary point $z \in E_{z_0}$ we can find the unique sequence $i_1, i_2, i_3, \dots$ for which

$$\forall \ell \in \mathbb{N} \quad : \quad z \in B_{i_1,\dots,i_\ell}.$$

We see that $|z - z_{i_1,\dots,i_\ell}| \leq \frac{1}{2} \cdot \operatorname{diam} B_{i_1,\dots,i_\ell} = r_\ell \underset{\ell \to \infty}{\longrightarrow} 0$, hence $z_{i_1,\dots,i_\ell} \underset{\ell \to \infty}{\longrightarrow} z$. Because by the construction $z_{i_1,\dots,i_\ell} \in E$ and the set E is compact, so we must have $z \in E$, which proves that $E_{z_0} \subset E$.

(ii)$\Longrightarrow$(iii) This implication is trivial, provided that $s \geq m$.

(iii)$\Longrightarrow$(iv) Assume that the set E is (m,s,κ)-perfect with constants $a,b>0$. By definition 4.5, for arbitrary $z_0 \in E$ and $0<r\le 1$ we can find an elementary (m,s,κ)-perfect set E_{z_0} such that $z_0 \in E_{z_0} \subset E$. We denote $E_{z_0} = \bigcap_{\ell=1}^{\infty} \bigcup_{i_1,\dots,i_\ell}^{1,\dots,\kappa} B_{i_1,\dots,i_\ell}$. By equality (3) we have $\operatorname{diam} B_{i_1,\dots,i_\ell} = a^{S(\ell)} \cdot (\operatorname{diam} B)^{m^\ell} = a^{S(\ell)}$, because $\operatorname{diam} B = 1$. We find the unique $\ell \in \mathbb{N}$ for which $a^{S(\ell)} < r \le a^{S(\ell-1)}$ and $i_1,\dots,i_\ell$ such that $z_0 \in B_{i_1,\dots,i_\ell}$. Exactly one of the balls $B_{i_1,\dots,i_\ell,j}$, where $j=1,\dots,\kappa$, contains the point z_0, say the last one. Consequently from each of the remaining balls we can select an arbitrary point $z_j \in E \cap B_{i_1,\dots,i_\ell,j}$, where $j=1,\dots,\kappa-1$, so that we have

$$z_j \in B_{i_1,\dots,i_\ell,j} \subset B_{i_1,\dots,i_\ell} \subset B\left(z_0, \operatorname{diam} B_{i_1,\dots,i_\ell}\right) = B\left(z_0, a^{S(\ell)}\right) \subset B(z_0,r),$$

$$\begin{aligned} |z_\mu - z_\nu| &\ge \operatorname{dist}\left(B_{i_1,\dots,i_\ell,\mu}, B_{i_1,\dots,i_\ell,\nu}\right) \ge a^{s\cdot S(\ell)} \cdot b = a^{s\cdot\left(m\cdot S(\ell-1)+1\right)} \cdot b = \\ &= a^s \cdot \left(a^{S(\ell-1)}\right)^{m\cdot s} \cdot b \ge a^s \cdot r^{m\cdot s} \cdot b \quad \text{for all } \mu,\nu = 1,\dots,\kappa \text{ such that } \mu \ne \nu, \end{aligned}$$

where we denote $z_\kappa := z_0$. This proves condition (iv) with the constant $c := a^s \cdot b$. $\square$

Corollary 4.8 [Białas-Eggink 1, theorem 3.4; Eggink, corollary 4.6]. *Any m-perfect set is $(m,2)$-perfect. Any $(m,2)$-perfect set is m^2-perfect.*

Proof. Definition 1.22 implies that a compact set $E \subset\subset \mathbb{C}$ is m-perfect if and only if [Siciak 2, proposition 0.1]

$$\exists 0<c\le 1 \quad \forall z_0 \in E \quad \forall 0<r\le 1 \quad \exists z_1 \in E \cap B(z_0,r) \quad : \quad |z_1 - z_0| \ge c\cdot r^m. \tag{8}$$

Therefore it is sufficient to apply proposition 4.7 with $s:=m$ and $\kappa := 2$. $\square$

Corollary 4.9 [Białas-Eggink 1, theorem 3.7]. *All m-perfect sets with $1 \le m < 2$ are L-regular.*

Proof. This is a direct consequence of corollary 4.8 and theorem 4.6. $\square$

Remark 4.10. If a compact set is m-perfect and hence $(m,2)$-perfect, where $m \ge 2$, while it is not (m,κ)-perfect with some $\kappa > m$, then it may have zero logarithmic capacity and consequently not be L-regular. Such an example of a Cantor-type set can be found in [Siciak 2, example 2.2].

A. Goncharov [Goncharov 2, corollary 3.1] also found $m=2$ to be the boundary value for the existence of a continuous and linear extension operator from the space of Whitney fields $\mathcal{E}(K)$ to $\mathcal{C}^\infty(\mathbb{R})$ for m-perfect Cantor-type sets $K \subset\subset \mathbb{R}$.

On the other hand, for any fixed $m>1$, A. Goncharov and H.B. Uzun [Goncharov-Uzun, example 2] have constructed a compact set on the real axis, which is m-perfect but not μ-perfect for any $\mu < m$, and it admits HCP($8m$).

Furthermore they have constructed an example [Goncharov-Uzun, example 1] of a set, which is m-perfect and actually also (m,∞)-perfect for any $m>1$, while it is not uniformly perfect and it does not admit GMI.

J. Lithner generalized the result of [Białas-Volberg] concerning the Cantor ternary set to prove the following theorem.

Theorem 4.11 [Lithner, theorem 5.1; cf. Siciak 5]. *For $0 < t \le \frac{1}{3}$, denote by $\mathcal{E}_t$ the family of elementary $(1,1,2)$-perfect sets with constants $a=b=t$ and $\operatorname{diam} B = 1$. Then all sets in this family admit* HCP *with constants $M,k \ge 1$ dependent only on t.*

Corollary 4.12. *Any uniformly perfect set admits* HCP.

Proof. Corollary 4.8, the first part of the proof of proposition 4.7 and definition 4.5 imply that any uniformly perfect set is the sum of elementary $(1,1,2)$-perfect sets belonging to the family $\mathcal{E}_t$, where $t := \frac{c}{8}$ depends only on the constant c in equality (8). This then leads to HCP by theorem 4.11. $\square$

Combining this with remark 3.5, we obtain the main result of [Lithner].

Corollary 4.13 [Lithner, theorem 6.2]. *If a compact set $E \subset\subset \mathbb{C}$ admits* WLMI(1)*, then it also admits* HCP *and* GMI.

Remark 4.14. L. Białas-Cież has proved in [Białas 1] the equivalence of the properties WLMI(1), HCP and GMI for all Cantor-type sets used by W. Pleśniak [Pleśniak 2].

THEOREM 4.15 [cf. Białas-Eggink 2, theorem 4.3]. *If a compact set $E \subset\subset \mathbb{C}$ admits* WLMP(m), *where $m \geq 1$, then it is an (m,∞)-perfect set.*

PROOF. We assume that

$$\forall n \in \mathbb{N} \quad \exists c_n \geq 1 \quad \forall z_0 \in E \quad \forall 0 < r \leq 1 \quad \forall p \in \mathcal{P}_n \quad \forall j = 1, \dots, n \quad : \\ |p^{(j)}(z_0)| \leq \left(\frac{c_n}{r^m}\right)^j \cdot \|p\|_{E \cap B(z_0,r)}.$$

Fix an arbitrary point $z_0 \in E$, $0 < r \leq 1$ and $\kappa \in \mathbb{N} \setminus \{1,2\}$. We will show that there exists a constant $0 < a_\kappa \leq 1$ dependent only on the set E and there exist points $z_1, \dots, z_{\kappa-1} \in E \cap B(z_0, r)$ such that

$$|z_\mu - z_\nu| \geq a_\kappa \cdot r^m \quad \text{for all } \mu, \nu = 0, \dots, \kappa - 1, \text{ such that } \mu \neq \nu,$$

which, according to proposition 4.7, is sufficient to prove that the set E is (m,κ)-perfect. Note that $(m,3)$-perfectness implies $(m,2)$-perfectness.

We put

$$a_\kappa := \frac{1}{2 \cdot (2e)^m \cdot (\kappa-1)^m \cdot c_{\kappa-1}}$$

and construct the points $\{z_j\}_{j=1,\dots,\kappa-1}$ as follows. Let $j := 1$.

Put

$$r_j := r \cdot \left(\frac{\kappa-2}{\kappa-1}\right)^{j-1}. \tag{9}$$

Now find κ Fekete extreme points for the intersection $E \cap B(z_0, r_j)$ and denote them by $\zeta_1^{(j)}, \dots, \zeta_\kappa^{(j)}$. These are distinct points because the set E is perfect. For $\mu = 1, \dots, \kappa$ denote by $L_{j,\mu} \in \mathcal{P}_{\kappa-1}$ the Lagrange polynomials (see definition 1.7)

$$L_{j,\mu}(z) := \frac{\prod\limits_{\substack{\ell=1,\dots,\kappa \\ \ell\neq\mu}} \left(z - \zeta_\ell^{(j)}\right)}{\prod\limits_{\substack{\ell=1,\dots,\kappa \\ \ell\neq\mu}} \left(\zeta_\mu^{(j)} - \zeta_\ell^{(j)}\right)}.$$

Observe that for all $1 \leq \mu < \nu \leq \kappa$ we have

$$\frac{1}{\left|\zeta_\nu^{(j)} - \zeta_\mu^{(j)}\right|^2} = \left| \frac{\prod\limits_{\substack{\ell=1,\dots,\kappa \\ \ell\neq\mu,\nu}} \left(\zeta_\nu^{(j)} - \zeta_\ell^{(j)}\right)}{\prod\limits_{\substack{\ell=1,\dots,\kappa \\ \ell\neq\mu}} \left(\zeta_\mu^{(j)} - \zeta_\ell^{(j)}\right)} \cdot \frac{\prod\limits_{\substack{\ell=1,\dots,\kappa \\ \ell\neq\mu,\nu}} \left(\zeta_\mu^{(j)} - \zeta_\ell^{(j)}\right)}{\prod\limits_{\substack{\ell=1,\dots,\kappa \\ \ell\neq\nu}} \left(\zeta_\nu^{(j)} - \zeta_\ell^{(j)}\right)} \right| = \left|\frac{d}{dz} L_{j,\mu}\left(\zeta_\nu^{(j)}\right)\right| \cdot \left|\frac{d}{dz} L_{j,\nu}\left(\zeta_\mu^{(j)}\right)\right|.$$

Thus, by the assumption, we have for each $0 < \varrho \leq 1$

$$\frac{1}{\left|\zeta_\nu^{(j)} - \zeta_\mu^{(j)}\right|^2} \leq \left(\frac{c_{\kappa-1}}{\varrho^m}\right)^2 \cdot \|L_{j,\mu}\|_{E \cap B(\zeta_\nu^{(j)},\varrho)} \cdot \|L_{j,\nu}\|_{E \cap B(\zeta_\mu^{(j)},\varrho)}.$$

Now assume that for each $\mu = 1, \dots, \kappa$ we have

$$\left|\zeta_\mu^{(j)} - z_0\right| \leq \frac{\kappa - 3/2}{\kappa - 1} \cdot r_j. \tag{10}$$

In such case we put $\varrho := \frac{1/2}{\kappa-1} \cdot r_j$ so that we have for all $\mu = 1, \dots, \kappa$

$$E \cap B(\zeta_\mu^{(j)}, \varrho) \subset E \cap B(z_0, r_j)$$

and for all $1 \le \mu < \nu \le \kappa$

$$\frac{1}{\left|\zeta_\nu^{(j)} - \zeta_\mu^{(j)}\right|^2} \le \left(\frac{c_{\kappa-1}}{\left(\frac{1/2}{\kappa-1}\cdot r_j\right)^m}\right)^2 \cdot \|L_{j,\mu}\|_{E\cap B(z_0,r_j)} \cdot \|L_{j,\nu}\|_{E\cap B(z_0,r_j)} = \left(\frac{2^m\cdot(\kappa-1)^m\cdot c_{\kappa-1}}{r_j^m}\right)^2$$

because by remark 1.8, the norm of these Lagrange polynomials on the set $E\cap B(z_0,r_j)$ is equal to 1. We see that

$$\left|\zeta_\nu^{(j)} - \zeta_\mu^{(j)}\right| \ge \frac{r_j^m}{2^m\cdot(\kappa-1)^m\cdot c_{\kappa-1}} = \frac{\left(r\cdot\left(\frac{\kappa-2}{\kappa-1}\right)^{j-1}\right)^m}{2^m\cdot(\kappa-1)^m\cdot c_{\kappa-1}}.$$

Note that for all $x>0$ we have $\left(\frac{x+1}{x}\right)^x = \left(1+\frac{1}{x}\right)^x < e$ and in particular, by putting $x := \kappa-2$ we see that $\left(\frac{\kappa-2}{\kappa-1}\right)^{\kappa-2} > \frac{1}{e}$. From this it follows that for all $1\le\mu<\nu\le\kappa$ we have

$$\left|\zeta_\nu^{(j)} - \zeta_\mu^{(j)}\right| \ge \frac{r^m}{(2e)^m\cdot(\kappa-1)^m\cdot c_{\kappa-1}} = 2a_\kappa\cdot r^m,$$

as long as $j\le\kappa-1$. Thus at most one point of the set $\{\zeta_\mu^{(j)}\}_{\mu=1,\dots,\kappa}$ can be included in the interior of the ball $B(z_0, a_\kappa\cdot r^m)$. After removing from $\{\zeta_\mu^{(j)}\}_{\mu=1,\dots,\kappa}$ that one point, or any arbitrary point if none belongs to the interior of $B(z_0,a_\kappa\cdot r^m)$, we are left with $\kappa-1$ points that meet the requirements of proposition 4.7.

If assumption (10) is not met, then we conclude that for a certain $\mu\in\{1,\dots,\kappa\}$ we have

$$\left|\zeta_\mu^{(j)} - z_0\right| > \frac{\kappa-3/2}{\kappa-1}\cdot r_j.$$

In this case we put $z_j := \zeta_\mu^{(j)}$, after which we increase j by 1 and return to (9).

This way, either for a certain $j\in\{1,\dots,\kappa-1\}$ condition (10) will be satisfied and then the problem will be solved, or we end up with a set of points $\{z_j\}_{j=1,\dots,\kappa-1}\subset E$ with the following property:

$$r\cdot\left(\frac{\kappa-2}{\kappa-1}\right)^{j-1}\cdot\frac{\kappa-3/2}{\kappa-1} = \frac{\kappa-3/2}{\kappa-1}\cdot r_j < |z_j-z_0| \le r_j = r\cdot\left(\frac{\kappa-2}{\kappa-1}\right)^{j-1}$$

for each $j\in\{1,\dots,\kappa-1\}$. From this it is easy to see that for all $1\le\mu<\nu\le\kappa-1$ we have

$$|z_\mu - z_\nu| \ge |z_\mu-z_0| - |z_\nu-z_0| \ge r\cdot\left(\frac{\kappa-2}{\kappa-1}\right)^{\mu-1}\cdot\frac{\kappa-3/2}{\kappa-1} - r\cdot\left(\frac{\kappa-2}{\kappa-1}\right)^{\nu-1} =$$
$$= r\cdot\left(\frac{\kappa-2}{\kappa-1}\right)^{\mu-1}\cdot\left(\frac{\kappa-3/2}{\kappa-1} - \left(\frac{\kappa-2}{\kappa-1}\right)^{\nu-\mu}\right) \ge r\cdot\left(\frac{\kappa-2}{\kappa-1}\right)^{\mu-1}\cdot\left(\frac{\kappa-3/2}{\kappa-1}-\frac{\kappa-2}{\kappa-1}\right) >$$
$$> r\cdot\frac{1}{e}\cdot\frac{1/2}{\kappa-1} \ge a_\kappa\cdot r^m$$

since $m\ge1$ and $c_\kappa\ge1$. But similarly we also have

$$|z_\mu - z_0| \ge r\cdot\left(\frac{\kappa-2}{\kappa-1}\right)^{\mu-1}\cdot\frac{\kappa-3/2}{\kappa-1} > r\cdot\frac{1}{e}\cdot\frac{1/2}{\kappa-1} \ge a_\kappa\cdot r^m$$

for each $\mu = 1,\dots,\kappa-1$, which finishes the proof. □

Corollary 4.16. *If a compact set $E\subset\subset\mathbb{C}$ admits* WLMP(m), *where $m\ge1$, then it admits* PP(m') *for any $m'>m^2$ if $m>1$ and $m'=1$ if $m=1$. Consequently by Wiener's criterion it is L-regular.*

Proof. By theorem 4.15, definition 4.5 and theorem 4.6, the set E admits PP$\left(\frac{m^2\cdot(\kappa-1)}{\kappa-m}\right)$ for any $\kappa\in\mathbb{N}$ such that $\kappa>m$. Now it suffices to note that $\lim_{\kappa\to\infty}\frac{m^2\cdot(\kappa-1)}{\kappa-m} = m^2$ and in the case that $m=1$, this limit is actually achieved for any κ.

Corollary 4.17. *If a compact set $E \subset\subset \mathbb{C}$ admits* LMP(m,k)*, where $m, k \geq 1$, implying that $c_n \leq c_1 \cdot n^k$ for all $n \in \mathbb{N}$, then in the proof of theorem 4.15 we can put*

$$a_\kappa := \frac{1}{2 \cdot (2e)^m \cdot c_1 \cdot \kappa^{m+k}}.$$

From the proof of proposition 4.7 and theorem 4.6 we obtain the following estimate for the logarithmic capacity for any $\kappa \in \mathbb{N}$ such that $\kappa > m$:

$$\forall z_0 \in E \quad \forall 0 < r \leq 1 \quad : \quad \operatorname{cap}\big(E \cap B(z_0, r)\big) \geq \left(\frac{C}{\kappa^{m+k}}\right)^{\frac{m \cdot \kappa}{\kappa - m} + 1} \cdot r^{\frac{m^2 \cdot (\kappa - 1)}{\kappa - m}}$$

where $C := \frac{1}{4 \cdot (8e)^m \cdot c_1}$ depends only on the set E.

Remark 4.18. L. Białas-Cież achieved a better estimate in [Białas 3, chapter II §2.2; see also Białas-Eggink 1, proposition 2.1] by using a different technique. She proved that for any compact set $E \subset\subset \mathbb{C}$ admitting LMP(m,k), where $m, k \geq 1$, we have

$$\forall z_0 \in E \quad \forall 0 < r \leq 1 \quad : \quad \operatorname{cap}\big(E \cap B(z_0, r)\big) \geq \frac{\vartheta^{2m+k}}{2^m \cdot c_1} \cdot r^m,$$

where

$$\vartheta := \exp\left(2 \cdot \sum_{j=2}^{\infty} \frac{\log(1/j)}{(j+1) \cdot (j+2)}\right) \approx \frac{3}{10}$$

is some absolute constant. This implies that the set E actually admits PP(m), however this technique does not give any geometric clues and also it does not work for WLMP.

CHAPTER V

SOBOLEV PROPERTY IN WHITNEY NORMS (SPW)

Definition 5.1 [cf. Bos-Milman §2]. For a smooth function $f \in \mathcal{C}^\infty(\mathbb{C})$, a compact set $E \subset\subset \mathbb{C}$ and $\ell \in \mathbb{N}$ we define:

$$|f|_{E,\ell} := \sum_{|\alpha|=\ell} \|D^\alpha f\|_E, \qquad |f|_{E,0} := \|f\|_E,$$

$$\|f\|_{E,\ell} := \|f\|_E + |f|_{E,\ell}, \qquad \|f\|_{E,0} := \|f\|_E,$$

$$T^\ell_{z_0} f(z) := \sum_{|\alpha|\le \ell-1} \frac{1}{\alpha!} \cdot D^\alpha f(z_0) \cdot (z-z_0)^\alpha,$$

$$R^\ell_{z_0} f(z) := f(z) - T^\ell_{z_0} f(z),$$

where for $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}^2_+$ we put:

$$|\alpha| := \alpha_1 + \alpha_2,$$

$$\alpha! := \alpha_1! \cdot \alpha_2!,$$

$$D^\alpha := \frac{\partial^{|\alpha|}}{\partial z^{\alpha_1} \cdot \partial \bar{z}^{\alpha_2}},$$

$$(z-z_0)^\alpha := (z-z_0)^{\alpha_1} \cdot (\bar{z} - \bar{z}_0)^{\alpha_2}.$$

$T^\ell_{z_0} f$ is the Taylor polynomial (not necessarily holomorphic) of the function f of degree $\ell - 1$ around the point $z_0 \in \mathbb{C}$ and $R^\ell_{z_0} f$ is its remainder.

Different versions of the following proposition are well known.

Proposition 5.2 Taylor formula with the remainder of Lagrange. *For any smooth function $f \in \mathcal{C}^\infty(\mathbb{C})$, $\ell \in \mathbb{N}$ and interval $I = [z_0, z_1] \subset \mathbb{C}$ we have*

$$|R^\ell_{z_0} f(z_1)| \le \min\left\{1, \frac{2^\ell}{\ell!}\right\} \cdot |z_1 - z_0|^\ell \cdot |f|_{I,\ell}.$$

Proof. Let's define two smooth functions for $t \in \mathbb{R}$

$$\widetilde{f}(t) := f\big(z_0 + t \cdot (z_1 - z_0)\big),$$

$$\varphi(t) := \sum_{j=0}^{\ell-1} \frac{1}{j!} \cdot (1-t)^j \cdot \widetilde{f}^{(j)}(t).$$

Then it follows that

$$\varphi'(t) = \sum_{j=1}^{\ell-1} \frac{1}{(j-1)!} \cdot (-1) \cdot (1-t)^{j-1} \cdot \widetilde{f}^{(j)}(t) + \sum_{j=0}^{\ell-1} \frac{1}{j!} \cdot (1-t)^j \cdot \widetilde{f}^{(j+1)}(t) =$$

$$= -\sum_{j=1}^{\ell-1} \frac{1}{(j-1)!} \cdot (1-t)^{j-1} \cdot \widetilde{f}^{(j)}(t) + \sum_{j=1}^{\ell} \frac{1}{(j-1)!} \cdot (1-t)^{j-1} \cdot \widetilde{f}^{(j)}(t) = \frac{1}{(\ell-1)!} \cdot (1-t)^{\ell-1} \cdot \widetilde{f}^{(\ell)}(t),$$

$$\varphi(1) - \varphi(0) = \int_0^1 \varphi'(t)\, dt,$$

$$|\varphi(1)-\varphi(0)| \le \int_0^1 |\varphi'(t)|\,dt \le \|\widetilde{f}^{(\ell)}\|_{[0,1]} \cdot \int_0^1 \frac{1}{(\ell-1)!}\cdot(1-t)^{\ell-1}\,dt = \frac{1}{\ell!}\cdot\|\widetilde{f}^{(\ell)}\|_{[0,1]}.$$

Differentiating $\widetilde{f}$ and using the Leibniz rule we obtain for each $j \in \mathbb{N}$

$$\widetilde{f}^{(j)}(t) = \sum_{|\alpha|=j} \binom{j}{\alpha_1} \cdot D^\alpha f\big(z_0 + t\cdot(z_1-z_0)\big)\cdot(z_1-z_0)^\alpha$$

and

$$\|\widetilde{f}^{(\ell)}\|_{[0,1]} \le \sum_{|\alpha|=\ell} \binom{\ell}{\alpha_1}\cdot\|D^\alpha f\|_I \cdot |z_1-z_0|^{|\alpha|} \le \max_{0\le\alpha_1\le\ell}\binom{\ell}{\alpha_1}\cdot|z_1-z_0|^\ell\cdot|f|_{I,\ell}.$$

On the other hand we have

$$\varphi(1) = \widetilde{f}(1) = f(z_1),$$

$$\varphi(0) = \sum_{j=0}^{\ell-1}\frac{1}{j!}\cdot\widetilde{f}^{(j)}(0) = \sum_{j=0}^{\ell-1}\sum_{|\alpha|=j}\frac{1}{\alpha_1!\cdot\alpha_2!}\cdot D^\alpha f(z_0)\cdot(z_1-z_0)^\alpha = T^\ell_{z_0}f(z_1).$$

Therefore we conclude that

$$\big|R^\ell_{z_0}f(z_1)\big| = \big|f(z_1)-T^\ell_{z_0}f(z_1)\big| = |\varphi(1)-\varphi(0)| \le \frac{1}{\ell!}\cdot\|\widetilde{f}^{(\ell)}\|_{[0,1]} \le$$
$$\le \frac{\max_{0\le\alpha_1\le\ell}\binom{\ell}{\alpha_1}}{\ell!}\cdot|z_1-z_0|^\ell\cdot|f|_{I,\ell} \le \min\left\{1,\frac{2^\ell}{\ell!}\right\}\cdot|z_1-z_0|^\ell\cdot|f|_{I,\ell}. \quad \square$$

Definition 5.3 [Eggink, definition 7.5; cf. Siciak 3]. For a compact set $E \subset\subset \mathbb{C}$ we define the family of smooth functions that are $\bar\partial$-flat on E:

$$\mathcal{A}^\infty(E) := \left\{\ f \in \mathcal{C}^\infty(\mathbb{C})\ :\ \text{the function } \frac{\partial f}{\partial\bar z} \text{ is flat on } E\ \right\}$$

A function $g \in \mathcal{C}^\infty(\mathbb{C})$ is said to be flat in the point z_0 if $D^\alpha g(z_0) = 0$ for all $\alpha \in \mathbb{Z}^2_+$. This definition is slightly different than in [Siciak 3], where $\mathcal{A}^\infty(E)$ stood for functions defined on E only, which will be denoted here as $\mathcal{A}^\infty(E)_{|E} := \{f_{|E}\ :\ f \in \mathcal{A}^\infty(E)\}$.

The following proposition is also well known to specialists.

Proposition 5.4. *If a compact set $E \subset\subset \mathbb{C}$ is perfect, then it is determining for functions of the class $\mathcal{A}^\infty(E)$, in other words $\mathcal{A}^\infty$-determining, which means:*

$$f \in \mathcal{A}^\infty(E),\ f_{|E} \equiv 0 \implies \forall\alpha \in \mathbb{Z}^2_+\ :\ (D^\alpha f)_{|E} \equiv 0.$$

Proof. Let's fix a function $f \in \mathcal{A}^\infty(E)$, such that $f \equiv 0$ on E, and a point $z_0 \in E$. Because the set E is perfect, we can find a sequence of different points $\{z_j\}_{j\in\mathbb{N}} \subset E$ such that $z_j \underset{j\to\infty}{\longrightarrow} z_0$ and next a subsequence also denoted $\{z_j\}_{j\in\mathbb{N}}$ such that $\arg(z_j - z_0) \underset{j\to\infty}{\longrightarrow} \gamma$ for some angle $\gamma \in [0, 2\pi)$. Then we have

$$\frac{f(z_j)-f(z_0)}{|z_j-z_0|} \underset{j\to\infty}{\longrightarrow} D_\gamma f(z_0),$$

where $D_\gamma f$ stands for the directional derivative of f, i.e.

$$D_\gamma f(z_0) := \lim_{t\to0}\frac{f(z_0+t\cdot e^{\gamma\cdot i})-f(z_0)}{t} = \cos\gamma\cdot\frac{\partial f}{\partial x}(z_0)+\sin\gamma\cdot\frac{\partial f}{\partial y}(z_0)$$

and, as usual, $z = x + y\cdot i$.

However, since $f(z_j) - f(z_0) = 0$ for all $j \in \mathbb{N}$, we see that

$$\cos\gamma\cdot\frac{\partial f}{\partial x}(z_0)+\sin\gamma\cdot\frac{\partial f}{\partial y}(z_0) = D_\gamma f(z_0) = 0,$$

while by the definition of $\mathcal{A}^\infty(E)$ we have

$$\frac{1}{2}\cdot\frac{\partial f}{\partial x}(z_0)+\frac{1}{2}\cdot i\cdot\frac{\partial f}{\partial y}(z_0)=\frac{\partial f}{\partial \bar z}(z_0)=0.$$

Solving this system of linear equations we obtain

$$\frac{\partial f}{\partial x}(z_0)=\frac{\partial f}{\partial y}(z_0)=0$$

and consequently $\frac{\partial f}{\partial z}(z_0)=0$, which means that $D^\alpha f\equiv 0$ on E if $|\alpha|=1$. We can now apply mathematical induction because $\frac{\partial f}{\partial z}\in\mathcal{A}^\infty(E)$. □

Definition 5.5 [cf. Bos-Milman, definition 2.1]. For a compact set $E\subset\subset\mathbb{C}$ and $\ell\in\mathbb{N}$ we define Whitney 'norms' for $f\in\mathcal{C}^\infty(\mathbb{C})$:

$$|||f|||_{E,\ell}:=\|f\|_E+\sup_{\substack{z,z_0\in E\\ z\neq z_0}}\frac{|R^\ell_{z_0}f(z)|}{|z-z_0|^\ell},$$

$$|||f|||_E:=\|f\|_E.$$

Remark 5.6. If $z,z_0\in E$, $z\neq z_0$, then by the Taylor formula with the remainder of Lagrange we have

$$\frac{|R^\ell_{z_0}f(z)|}{|z-z_0|^\ell}\le\min\left\{1,\frac{2^\ell}{\ell!}\right\}\cdot|f|_{[z_0,z],\ell}\le\min\left\{1,\frac{2^\ell}{\ell!}\right\}\cdot|f|_{\mathrm{conv}\,E,\ell},$$

where $\mathrm{conv}\,E$ stands for the convex hull of the set E. This shows that the Whitney norms are well defined.

On the other hand, if we assume that the set E is perfect and thus $\mathcal{A}^\infty$-determining and also that $f\in\mathcal{A}^\infty(E)$, then we see that

$$R^\ell_{z_0}f(z)=f(z)-T^\ell_{z_0}f(z)=\sum_{|\alpha|=\ell}\frac{1}{\alpha!}\cdot D^\alpha f(z_0)\cdot(z-z_0)^\alpha+O\left(|z-z_0|^{\ell+1}\right)=$$

$$=\frac{1}{\ell!}\cdot f^{(\ell)}(z_0)\cdot(z-z_0)^\ell+O\left(|z-z_0|^{\ell+1}\right).$$

In this case we obtain

$$\sup_{\substack{z\in E\\ z\neq z_0}}\frac{|R^\ell_{z_0}f(z)|}{|z-z_0|^\ell}\ge\lim_{z\to z_0}\frac{|R^\ell_{z_0}f(z)|}{|z-z_0|^\ell}=\frac{1}{\ell!}\cdot|f^{(\ell)}(z_0)|,$$

$$\sup_{\substack{z,z_0\in E\\ z\neq z_0}}\frac{|R^\ell_{z_0}f(z)|}{|z-z_0|^\ell}\ge\sup_{z_0\in E}\frac{1}{\ell!}\cdot|f^{(\ell)}(z_0)|=\frac{1}{\ell!}\cdot||f^{(\ell)}||_E=\frac{1}{\ell!}\cdot|f|_{E,\ell}.$$

Remark 5.7. The Whitney 'norms' are in the general case only seminorms. If a compact set E is $\mathcal{C}^\infty$-determining (respectively $\mathcal{A}^\infty$-determining), then any function $f\in\mathcal{C}^\infty(E):=\mathcal{C}^\infty(\mathbb{C})_{|E}$ (respectively $f\in\mathcal{A}^\infty(E)_{|E}$) can be identified with its Whitney field on the set E, i.e. $(D^\alpha f)_{\alpha\in\mathbb{Z}^2_+}$, which in turn determines the Taylor polynomials $T^\ell_{z_0}$ for any $z_0\in E$. In such case the Whitney norms are norms on the space $\mathcal{E}(E)$ of Whitney fields or on $\mathcal{C}^\infty(E)$ (respectively $\mathcal{A}^\infty(E)_{|E}$). Note also that many different versions of the Whitney norms appear in the literature, see e.g. [Whitney], [Tidten 1], [Tidten 2], [Bos-Milman 1] or [Bos-Milman 3].

Definition 5.8 [cf. Eggink, definition 7.8; cf. Bos-Milman, definition 2.12]. A compact set $E\subset\subset\mathbb{C}$ admits the Sobolev Property in Whitney norms $\mathrm{SPW}(m,k)$, where $m,k\ge1$, if

$$\forall\ell\in\mathbb{N}\quad\exists c_\ell\ge1\quad\forall j\in\mathbb{N}\text{ such that }\ell\ge m\cdot j\quad\forall f\in\mathcal{A}^\infty(E)\quad:$$
$$|f|_{E,j}\le c_\ell^j\cdot|||f|||_E^{1-\frac{m\cdot j}{\ell}}\cdot|||f|||_{E,\ell}^{\frac{m\cdot j}{\ell}}$$

and additionally $c_\ell\le c_1\cdot\ell^k$. Without the last assumption we speak of the Weak Sobolev Property in Whitney norms $\mathrm{WSPW}(m)$.

Theorem 5.9 [cf. Eggink, theorem 7.9; cf. Bos-Milman, theorem A]. *For any compact set $E \subset\subset \mathbb{C}$ and $m, k \geq 1$ we have*

$$\mathrm{LMP}(m,k) \implies \mathrm{SPW}(m,k),$$
$$\mathrm{WLMP}(m) \implies \mathrm{WSPW}(m).$$

Proof. Let's first assume that the set E admits WLMP(m), i.e.

$$\forall n \in \mathbb{N} \quad \exists c_n \geq 1 \quad \forall z_0 \in E \quad \forall 0 < r \leq 1 \quad \forall p \in \mathcal{P}_n \quad \forall j = 1, \dots, n \quad : \quad |p^{(j)}(z_0)| \leq \left(\frac{c_n}{r^m}\right)^j \cdot \|p\|_{E\cap B(z_0,r)}.$$

Without loss of generality we can assume that $c_n \geq n$. By remark 3.5 and proposition 5.4 the set E is m-perfect and $\mathcal{A}^\infty$-determining. Fix $f \in \mathcal{A}^\infty(E)$ and assume that $f_{|E} \not\equiv 0$, since otherwise we would have $|f|_{E,j} = 0$ for all $j \in \mathbb{N}$ and the assertion would be fulfilled. For arbitrary $z_0 \in E$, $0 < r \leq 1$ and $\ell \in \mathbb{N}$ we have

$$\|R^\ell_{z_0} f\|_{E\cap B(z_0,r)} = \sup_{z\in E\cap B(z_0,r)} |R^\ell_{z_0} f(z)| = \sup_{\substack{z\in E\cap B(z_0,r)\\ z\neq z_0}} |R^\ell_{z_0} f(z)| \leq$$

$$\leq r^\ell \cdot \sup_{\substack{z\in E\cap B(z_0,r)\\ z\neq z_0}} \frac{|R^\ell_{z_0} f(z)|}{|z-z_0|^\ell} \leq r^\ell \cdot \sup_{\substack{a,z\in E\\ a\neq z}} \frac{|R^\ell_a f(z)|}{|z-a|^\ell} = r^\ell \cdot \left(|||f|||_{E,\ell} - \|f\|_E\right) \leq r^\ell \cdot |||f|||_{E,\ell},$$

and therefore

$$\|T^\ell_{z_0} f\|_{E\cap B(z_0,r)} = \|f - R^\ell_{z_0} f\|_{E\cap B(z_0,r)} \leq \|f\|_{E\cap B(z_0,r)} + \|R^\ell_{z_0} f\|_{E\cap B(z_0,r)} \leq \|f\|_E + r^\ell \cdot |||f|||_{E,\ell}.$$

For all $\alpha \in \mathbb{Z}^2_+$ such that $|\alpha| \leq \ell - 1$ we have $D^\alpha f(z_0) = D^\alpha(T^\ell_{z_0} f)(z_0)$ because $T^\ell_{z_0} f$ is the Taylor polynomial of function f of degree $\ell - 1$ at the point z_0. Since $f \in \mathcal{A}^\infty(E)$, we have $D^\alpha f(z_0) = 0$ for all α such that $\alpha_2 \geq 1$ and consequently $T^\ell_{z_0} f \in \mathcal{P}_{\ell-1}$ is a holomorphic polynomial. We can therefore apply WLMP to obtain for $j = 1, \dots, \ell - 1$

$$\left|\frac{\partial^j f}{\partial z^j}(z_0)\right| = \left|(T^\ell_{z_0} f)^{(j)}(z_0)\right| \leq \left(\frac{c_\ell}{r^m}\right)^j \cdot \|T^\ell_{z_0} f\|_{E\cap B(z_0,r)} \leq \left(\frac{c_\ell}{r^m}\right)^j \cdot \left(\|f\|_E + r^\ell \cdot |||f|||_{E,\ell}\right).$$

This estimate is also true for $j = \ell$ because by remark 5.6 we have

$$\left|\frac{\partial^\ell f}{\partial z^\ell}(z_0)\right| \leq \sum_{|\alpha|=\ell} \|D^\alpha f\|_E = |f|_{E,\ell} \leq \ell! \cdot |||f|||_{E,\ell} \leq \left(\frac{\ell \cdot r}{r^m}\right)^\ell \cdot |||f|||_{E,\ell} \leq \left(\frac{c_\ell \cdot r}{r^m}\right)^\ell \cdot |||f|||_{E,\ell}.$$

We put

$$r := \left(\frac{\|f\|_E}{|||f|||_{E,\ell}}\right)^{1/\ell} \leq 1$$

to see that

$$\left|\frac{\partial^j f}{\partial z^j}(z_0)\right| \leq \left(c_\ell \cdot \left(\frac{|||f|||_{E,\ell}}{\|f\|_E}\right)^{m/\ell}\right)^j \cdot \left(\|f\|_E + \frac{\|f\|_E}{|||f|||_{E,\ell}} \cdot |||f|||_{E,\ell}\right) = c_\ell^j \cdot |||f|||_{E,\ell}^{\frac{m\cdot j}{\ell}} \cdot \|f\|_E^{1-\frac{m\cdot j}{\ell}} \cdot 2.$$

Because the point $z_0 \in E$ was arbitrary, we obtain for all $\ell \in \mathbb{N}$ and $j = 1, \dots, \ell$

$$|f|_{E,j} = \sum_{|\alpha|=j} \|D^\alpha f\|_E = \left\|\frac{\partial^j f}{\partial z^j}\right\|_E \leq \widetilde{c}_\ell^{\,j} \cdot |||f|||_E^{1-\frac{m\cdot j}{\ell}} \cdot |||f|||_{E,\ell}^{\frac{m\cdot j}{\ell}},$$

where $\widetilde{c}_\ell := 2c_\ell$. This finishes the proof of WSPW(m), but obviously if $c_n \leq c_1 \cdot n^k$ for all $n \in \mathbb{N}$ then also $\widetilde{c}_\ell \leq 2c_1 \cdot \ell^k = \widetilde{c}_1 \cdot \ell^k$, as required in SPW$(m,k)$. □

CHAPTER VI

SOBOLEV PROPERTY IN QUOTIENT NORMS (SPQ)

Definition 6.1 [Białas-Eggink 2, definition 1.4; Eggink, definition 8.1; cf. Bos-Milman, definition 2.2]. For a compact set $E \subset\subset \mathbb{C}$ and $\ell \in \mathbb{N}$ we define quotient norms for $f \in \mathcal{A}^\infty(E)$ (or as the case may be $f \in \mathcal{A}^\infty(E)_{|E}$):

$$\bracevert f \bracevert_{E,\ell} := \inf\left\{ \|\widetilde{f}\|_{\operatorname{conv} E,\ell} : \ \widetilde{f} \in \mathcal{A}^\infty(E), \ \widetilde{f}_{|E} \equiv f_{|E} \right\},$$

$$\bracevert f \bracevert_{E} := \|f\|_E.$$

Definition 6.2 [cf. Białas-Eggink 2, definition 1.5; cf. Eggink, definition 8.2; cf. Bos-Milman, definition 2.15]. A compact set $E \subset\subset \mathbb{C}$ admits the Sobolev Property in Quotient norms $\mathrm{SPQ}(m,k)$, where $m, k \geq 1$, if

$$\forall \ell \in \mathbb{N} \quad \exists c_\ell \geq 1 \quad \forall j \in \mathbb{N} \text{ such that } \ell \geq m \cdot j \quad \forall f \in \mathcal{A}^\infty(E) \quad :$$
$$|f|_{E,j} \leq c_\ell^j \cdot \bracevert f \bracevert_E^{1-\frac{m\cdot j}{\ell}} \cdot \bracevert f \bracevert_{E,\ell}^{\frac{m\cdot j}{\ell}}$$

and additionally $c_\ell \leq c_1 \cdot \ell^k$. Without the last assumption we speak of the Weak Sobolev Property in Quotient norms $\mathrm{WSPQ}(m)$.

Theorem 6.3 [cf. Eggink, theorem 8.3; cf. Bos-Milman, theorem A]. *For any compact set $E \subset\subset \mathbb{C}$ and $m, k \geq 1$ we have*

$$\mathrm{SPW}(m,k) \implies \mathrm{SPQ}(m,k),$$
$$\mathrm{WSPW}(m) \implies \mathrm{WSPQ}(m).$$

Proof. Let's first assume that the set E admits $\mathrm{WSPW}(m)$, i.e.

$$\forall \ell \in \mathbb{N} \quad \exists c_\ell \geq 1 \quad \forall j \in \mathbb{N} \text{ such that } \ell \geq m \cdot j \quad \forall f \in \mathcal{A}^\infty(E) \quad :$$
$$|f|_{E,j} \leq c_\ell^j \cdot |||f|||_E^{1-\frac{m\cdot j}{\ell}} \cdot |||f|||_{E,\ell}^{\frac{m\cdot j}{\ell}}.$$

Fix ℓ, j and f as above, and let's take an arbitrary $\widetilde{f} \in \mathcal{A}^\infty(E)$ such that $\widetilde{f}_{|E} = f_{|E}$. By applying WSPW to $(\widetilde{f} - f)$ we see that the set E is $\mathcal{A}^\infty$-determining and for all $\alpha \in \mathbb{Z}_+^2$ we have $D^\alpha \widetilde{f} \equiv D^\alpha f$ on E. Therefore by the Taylor formula with the remainder of Lagrange we obtain as in remark 5.6

$$|||f|||_{E,\ell} = \|f\|_E + \sup_{\substack{z,z_0 \in E \\ z \neq z_0}} \frac{|R^\ell_{z_0} f(z)|}{|z - z_0|^\ell} = \|\widetilde{f}\|_E + \sup_{\substack{z,z_0 \in E \\ z \neq z_0}} \frac{|R^\ell_{z_0} \widetilde{f}(z)|}{|z - z_0|^\ell} \leq \|\widetilde{f}\|_{\operatorname{conv} E} + |\widetilde{f}|_{\operatorname{conv} E,\ell} = \|\widetilde{f}\|_{\operatorname{conv} E,\ell}.$$

Taking the infimum over such $\widetilde{f}$ we obtain

$$|||f|||_{E,\ell} \leq \inf\left\{ \|\widetilde{f}\|_{\operatorname{conv} E,\ell} \ : \ \widetilde{f} \in \mathcal{A}^\infty(E), \ \widetilde{f}_{|E} \equiv f_{|E} \right\} = \bracevert f \bracevert_{E,\ell},$$

and putting this into the inequality WSPW we conclude that

$$|f|_{E,j} \leq c_\ell^j \cdot \bracevert f \bracevert_E^{1-\frac{m\cdot j}{\ell}} \cdot \bracevert f \bracevert_{E,\ell}^{\frac{m\cdot j}{\ell}}.$$

This finishes the proof of $\mathrm{WSPQ}(m)$, respectively $\mathrm{SPQ}(m,k)$ if $c_\ell \leq c_1 \cdot \ell^k$ for all $\ell \in \mathbb{N}$. $\square$

Definition 6.4 [Białas-Eggink 2, definition 1.7]. For a compact set $E \subset\subset \mathbb{C}$ we define the family of smooth functions that are holomorphic in some open neighbourhood of the set E:

$$\mathcal{H}^\infty(E) := \left\{ f \in \mathcal{C}^\infty(\mathbb{C}) \ : \ \frac{\partial f}{\partial \bar{z}} \equiv 0 \text{ in some open neighbourhood of } E \right\}.$$

Definition 6.5 [Białas-Eggink 2, definition 1.7]. For a compact set $E \subset\subset \mathbb{C}$ and $\ell \in \mathbb{N}$ we define holomorphic quotient norms for $f \in \mathcal{H}^\infty(E)$:

$$\langle\!\langle f \rangle\!\rangle_{E,\ell} := \inf \left\{ \|\widetilde{f}\|_{\operatorname{conv} E,\ell} : \ \widetilde{f} \in \mathcal{H}^\infty(E), \ \widetilde{f}_{|E} \equiv f_{|E} \right\},$$

$$\langle\!\langle f \rangle\!\rangle_{E} := \|f\|_E.$$

Definition 6.6 [cf. Białas-Eggink 2, definition 1.7]. A compact set $E \subset\subset \mathbb{C}$ admits the Sobolev Property in Quotient norms for Holomorphic functions SPQH(m,s,k) where $m,k \geq 1$ and $s \geq 0$, if

$$\forall \ell \in \mathbb{N} \quad \exists c_\ell \geq 1 \quad \forall j \in \mathbb{N} \text{ such that } \ell \geq m \cdot j \quad \forall 0 < \delta \leq 1 \quad \forall f \in \mathcal{H}^\infty(E_\delta) \quad :$$
$$|f|_{E,j} \leq \left(\frac{c_\ell}{\delta^s}\right)^j \cdot \langle\!\langle f \rangle\!\rangle_E^{1-\frac{m\cdot j}{\ell}} \cdot \langle\!\langle f \rangle\!\rangle_{E_\delta,\ell}^{\frac{m\cdot j}{\ell}}$$

and additionally $c_\ell \leq c_1 \cdot \ell^k$. Without the last assumption we speak of the Weak Sobolev Property in Quotient norms for Holomorphic functions WSPQH(m,s).

Theorem 6.7 [cf. Białas-Eggink 2, theorem 1.8c]. *For any compact set $E \subset\subset \mathbb{C}$, $m,k \geq 1$ and any $s \geq 0$ we have*

$$\text{SPQ}(m,k) \implies \text{SPQH}(m,s,k),$$
$$\text{WSPQ}(m) \implies \text{WSPQH}(m,s).$$

Proof. The proof is immediate because $\mathcal{H}^\infty(E_\delta) \subset \mathcal{H}^\infty(E) \subset \mathcal{A}^\infty(E)$ and $\operatorname{conv} E \subset \operatorname{conv} E_\delta$. Consequently for all $\ell \in \mathbb{N}$, $0 < \delta \leq 1$ and $f \in \mathcal{H}^\infty(E_\delta)$ we have $\|\!| f |\!\|_{E,\ell} \leq \langle\!\langle f \rangle\!\rangle_{E,\ell} \leq \langle\!\langle f \rangle\!\rangle_{E_\delta,\ell}$. $\square$

While the implication SPQH $\implies$ LMP was the main subject of [Białas-Eggink 2], here we will produce a more convenient result by introducing yet another Sobolev property. First however, following the example of [Bos-Milman], we construct special cutoff functions, which will allow us to estimate the holomorphic quotient norms.

Proposition 6.8 on cutoff functions [cf. Bos-Milman, lemma 4.12; cf. Tougeron, lemma 3.3; cf. Malgrange, lemma 4.2]. *For any compact set $K \subset\subset \mathbb{C}$ and radius $0 < \epsilon \leq 1$ there exists a cutoff function $u \in \mathcal{C}^\infty(\mathbb{C})$ such that*

$$\begin{array}{lll} (a) & 0 \leq u(z) \leq 1 & \textit{for all } z \in \mathbb{C}, \\ (b) & u(z) = 1 & \textit{if } \operatorname{dist}(z,K) \leq \dfrac{\epsilon}{8}, \\ (c) & u(z) = 0 & \textit{if } \operatorname{dist}(z,K) \geq \epsilon, \\ (d) & \|D^\alpha u\|_{\mathbb{C}} \leq \dfrac{C_{|\alpha|}}{\epsilon^{|\alpha|}} & \textit{for all } \alpha = (\alpha_1,\alpha_2) \in \mathbb{Z}_+^2 \end{array}$$

where $C_t := d \cdot t^{4t}$ for $t \in \mathbb{N}$, $C_0 := d$ and $d \geq 1$ is some absolute constant.

In order to prove this proposition we need two auxiliary lemmas.

Lemma 6.9. *Let $\chi \in \mathcal{C}^\infty(\mathbb{R})$ and $a \in \mathbb{R}$. Put $\upsilon(x) := \chi\left(x^2 + a\right)$ for $x \in \mathbb{R}$ so that $\upsilon \in \mathcal{C}^\infty(\mathbb{R})$. Then for $\mu \in \mathbb{Z}_+$ we have*

$$(1) \qquad \begin{aligned} \upsilon^{(2\mu)}(x) &= \sum_{\ell=\mu}^{2\mu} a_{2\mu,\ell} \cdot x^{2\ell-2\mu} \cdot \chi^{(\ell)}\left(x^2+a\right), \\ \upsilon^{(2\mu+1)}(x) &= \sum_{\ell=\mu+1}^{2\mu+1} a_{2\mu+1,\ell} \cdot x^{2\ell-2\mu-1} \cdot \chi^{(\ell)}\left(x^2+a\right). \end{aligned}$$

where for all $\mu \in \mathbb{N}$

$$\begin{aligned} a_{2\mu,2\mu} &= 2a_{2\mu-1,2\mu-1}, \\ a_{2\mu,\ell} &= a_{2\mu-1,\ell}\cdot(2\ell-2\mu+1)+2a_{2\mu-1,\ell-1} \qquad \text{for all } \ell\in\mathbb{N} \text{ such that } \mu<\ell<2\mu, \\ a_{2\mu,\mu} &= a_{2\mu-1,\mu} \end{aligned}$$

and for all $\mu \in \mathbb{Z}_+$

$$\begin{aligned} a_{2\mu+1,2\mu+1} &= 2a_{2\mu,2\mu}, \\ a_{2\mu+1,\ell} &= a_{2\mu,\ell}\cdot(2\ell-2\mu)+2a_{2\mu,\ell-1} \qquad \text{for all } \ell\in\mathbb{N} \text{ such that } \mu<\ell<2\mu+1. \end{aligned}$$

In particular, for $\nu \in \mathbb{Z}_+$ *and* $\ell = \operatorname{int}\frac{\nu+1}{2},\dots,\nu$ *we have* $1 \le a_{\nu,\ell} \le (\nu+1)!$ *and consequently*

$$\upsilon^{(\nu)}(x) = \sum_{\ell=\operatorname{int}\frac{\nu+1}{2}}^{\nu} a_{\nu,\ell}\cdot x^{2\ell-\nu}\cdot\chi^{(\ell)}\left(x^2+a\right), \tag{2}$$

$$|\upsilon^{(\nu)}(x)| \le (\nu+1)!\cdot\max\{1,|x|^{\nu}\}\cdot\sum_{\ell=\operatorname{int}\frac{\nu+1}{2}}^{\nu}\left|\chi^{(\ell)}\left(x^2+a\right)\right|.$$

Here and further $\operatorname{int}\frac{\nu+1}{2}$ *denotes the largest integer smaller than or equal to* $\frac{\nu+1}{2}$.

Proof. Obviously for $\mu = 0$ we see that equalities (1) are true with $a_{0,0} = 1$ and $a_{1,1} = 2$ since $\upsilon'(x) = 2x\cdot\chi'\left(x^2+a\right)$. Using mathematical induction we can prove equalities (1) for any $\mu\in\mathbb{N}$ because

$$\begin{aligned} \upsilon^{(2\mu)}(x) &= \left(\upsilon^{(2(\mu-1)+1)}\right)'(x) = \\ &= \sum_{\ell=\mu}^{2\mu-1} a_{2\mu-1,\ell}\cdot\Big((2\ell-2\mu+1)\cdot x^{2\ell-2\mu}\cdot\chi^{(\ell)}\left(x^2+a\right)+x^{2\ell-2\mu+1}\cdot 2x\cdot\chi^{(\ell+1)}\left(x^2+a\right)\Big) = \\ &= \sum_{\ell=\mu}^{2\mu-1} a_{2\mu-1,\ell}\cdot(2\ell-2\mu+1)\cdot x^{2\ell-2\mu}\cdot\chi^{(\ell)}\left(x^2+a\right)+\sum_{\ell=\mu}^{2\mu-1} 2a_{2\mu-1,\ell}\cdot x^{2\ell-2\mu+2}\cdot\chi^{(\ell+1)}\left(x^2+a\right) = \\ &= \sum_{\ell=\mu}^{2\mu-1} a_{2\mu-1,\ell}\cdot(2\ell-2\mu+1)\cdot x^{2\ell-2\mu}\cdot\chi^{(\ell)}\left(x^2+a\right)+\sum_{\ell=\mu+1}^{2\mu} 2a_{2\mu-1,\ell-1}\cdot x^{2\ell-2\mu}\cdot\chi^{(\ell)}\left(x^2+a\right) = \\ &= \sum_{\ell=\mu}^{2\mu} a_{2\mu,\ell}\cdot x^{2\ell-2\mu}\cdot\chi^{(\ell)}\left(x^2+a\right) \end{aligned}$$

and

$$\begin{aligned} \upsilon^{(2\mu+1)}(x) &= \left(\upsilon^{(2\mu)}\right)'(x) = \\ &= \sum_{\ell=\mu}^{2\mu} a_{2\mu,\ell}\cdot\Big((2\ell-2\mu)\cdot x^{2\ell-2\mu-1}\cdot\chi^{(\ell)}\left(x^2+a\right)+x^{2\ell-2\mu}\cdot 2x\cdot\chi^{(\ell+1)}\left(x^2+a\right)\Big) = \\ &= \sum_{\ell=\mu}^{2\mu} a_{2\mu,\ell}\cdot(2\ell-2\mu)\cdot x^{2\ell-2\mu-1}\cdot\chi^{(\ell)}\left(x^2+a\right)+\sum_{\ell=\mu}^{2\mu} 2a_{2\mu,\ell}\cdot x^{2\ell-2\mu+1}\cdot\chi^{(\ell+1)}\left(x^2+a\right) = \\ &= \sum_{\ell=\mu+1}^{2\mu} a_{2\mu,\ell}\cdot(2\ell-2\mu)\cdot x^{2\ell-2\mu-1}\cdot\chi^{(\ell)}\left(x^2+a\right)+\sum_{\ell=\mu+1}^{2\mu+1} 2a_{2\mu,\ell-1}\cdot x^{2\ell-2\mu-1}\cdot\chi^{(\ell)}\left(x^2+a\right) = \\ &= \sum_{\ell=\mu+1}^{2\mu+1} a_{2\mu+1,\ell}\cdot x^{2\ell-2\mu-1}\cdot\chi^{(\ell)}\left(x^2+a\right). \end{aligned}$$

It is now obvious that $a_{\nu,\ell} \ge 1$ for all $\nu\in\mathbb{Z}_+$ and $\ell = \operatorname{int}\frac{\nu+1}{2},\dots,\nu$.

Finally put $b_\nu := \max_\ell a_{\nu,\ell}$ for all $\nu\in\mathbb{N}$. Because for all $\mu\in\mathbb{N}$ and eligible ℓ we have $a_{2\mu,\ell} \le (2\mu+1)\cdot b_{2\mu-1}$ and $a_{2\mu+1,\ell} \le (2\mu+2)\cdot b_{2\mu}$, we see that $b_{2\mu} \le (2\mu+1)\cdot b_{2\mu-1}$ and $b_{2\mu+1} \le (2\mu+2)\cdot b_{2\mu}$ so that altogether $b_\nu \le (\nu+1)!$. □

COROLLARY 6.10. *Let $\chi \in \mathcal{C}^\infty(\mathbb{R})$ and $b \in \mathbb{R}$. Put $\Upsilon(\xi) := \chi\left(\|\xi\|^2 - b\right)$ for $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$ so that $\Upsilon \in \mathcal{C}^\infty(\mathbb{R}^2)$. Here $\|\xi\| := \sqrt{\xi_1^2 + \xi_2^2}$ stands for the standard Euclidean norm. Then for any $\xi \in \mathbb{R}^2$ and $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}_+^2$ such that $|\alpha| > 0$, we have*

$$\left|\frac{\partial^{|\alpha|}}{\partial\xi_1^{\alpha_1} \cdot \partial\xi_2^{\alpha_2}}\Upsilon(\xi)\right| \le (\alpha_1+1)! \cdot (\alpha_2+1)! \cdot \max\left\{1, \|\xi\|^{|\alpha|}\right\} \cdot |\alpha| \cdot \sum_{\ell=\operatorname{int}\frac{|\alpha|+1}{2}}^{|\alpha|} \left|\chi^{(\ell)}\left(\|\xi\|^2 - b\right)\right|.$$

PROOF. We apply equality (2) of lemma 6.9 twice; first to the function $\Upsilon(\,\cdot\,, \xi_2)$ with constant ξ_2, i.e. $\upsilon(x) := \chi\left(x^2 + a\right)$ where $a := \xi_2^2 - b$, to obtain

$$\frac{\partial^{\alpha_1}}{\partial\xi_1^{\alpha_1}}\Upsilon(\xi_1, \xi_2) = \sum_{\ell_1=\operatorname{int}\frac{\alpha_1+1}{2}}^{\alpha_1} a_{\alpha_1,\ell_1} \cdot \xi_1^{2\ell_1-\alpha_1} \cdot \chi^{(\ell_1)}\left(\xi_1^2 + \xi_2^2 - b\right),$$

and next to the constituents of the function $\frac{\partial^{\alpha_1}}{\partial\xi_1^{\alpha_1}}\Upsilon(\xi_1, \,\cdot\,)$ with constant ξ_1, i.e. $\upsilon(x) := \chi^{(\ell_1)}\left(x^2 + a\right)$ where $a := \xi_1^2 - b$, to see that

$$\frac{\partial^{|\alpha|}}{\partial\xi_1^{\alpha_1} \cdot \partial\xi_2^{\alpha_2}}\Upsilon(\xi) = \frac{\partial^{\alpha_2}}{\partial\xi_2^{\alpha_2}}\left(\frac{\partial^{\alpha_1}}{\partial\xi_1^{\alpha_1}}\Upsilon\right)(\xi_1, \xi_2) =$$
$$= \sum_{\ell_1=\operatorname{int}\frac{\alpha_1+1}{2}}^{\alpha_1} a_{\alpha_1,\ell_1} \cdot \xi_1^{2\ell_1-\alpha_1} \cdot \left(\sum_{\ell_2=\operatorname{int}\frac{\alpha_2+1}{2}}^{\alpha_2} a_{\alpha_2,\ell_2} \cdot \xi_2^{2\ell_2-\alpha_2} \cdot \chi^{(\ell_1+\ell_2)}\left(\xi_1^2 + \xi_2^2 - b\right)\right).$$

We conclude that

$$\begin{aligned}
&\left|\frac{\partial^{|\alpha|}}{\partial\xi_1^{\alpha_1} \cdot \partial\xi_2^{\alpha_2}}\Upsilon(\xi)\right| \le \\
&\le \sum_{\ell_1=\operatorname{int}\frac{\alpha_1+1}{2}}^{\alpha_1} \sum_{\ell_2=\operatorname{int}\frac{\alpha_2+1}{2}}^{\alpha_2} a_{\alpha_1,\ell_1} \cdot a_{\alpha_2,\ell_2} \cdot |\xi_1|^{2\ell_1-\alpha_1} \cdot |\xi_2|^{2\ell_2-\alpha_2} \cdot \left|\chi^{(\ell_1+\ell_2)}\left(\|\xi\|^2 - b\right)\right| \le \\
&\le \sum_{\ell_1=\operatorname{int}\frac{\alpha_1+1}{2}}^{\alpha_1} \sum_{\ell_2=\operatorname{int}\frac{\alpha_2+1}{2}}^{\alpha_2} (\alpha_1+1)! \cdot (\alpha_2+1)! \cdot \|\xi\|^{2\ell_1-\alpha_1+2\ell_2-\alpha_2} \cdot \left|\chi^{(\ell_1+\ell_2)}\left(\|\xi\|^2 - b\right)\right| \le \\
&\le \sum_{\ell_1=\operatorname{int}\frac{\alpha_1+1}{2}}^{\alpha_1} \sum_{\ell_2=\operatorname{int}\frac{\alpha_2+1}{2}}^{\alpha_2} (\alpha_1+1)! \cdot (\alpha_2+1)! \cdot \max\left\{1, \|\xi\|^{|\alpha|}\right\} \cdot \left|\chi^{(\ell_1+\ell_2)}\left(\|\xi\|^2 - b\right)\right| \le \\
&\le (\alpha_1+1)! \cdot (\alpha_2+1)! \cdot \max\left\{1, \|\xi\|^{|\alpha|}\right\} \cdot \sum_{\ell_1=\operatorname{int}\frac{\alpha_1+1}{2}}^{\alpha_1} \sum_{\ell=\operatorname{int}\frac{|\alpha|+1}{2}}^{|\alpha|} \left|\chi^{(\ell)}\left(\|\xi\|^2 - b\right)\right| \le \\
&\le (\alpha_1+1)! \cdot (\alpha_2+1)! \cdot \max\left\{1, \|\xi\|^{|\alpha|}\right\} \cdot |\alpha| \cdot \sum_{\ell=\operatorname{int}\frac{|\alpha|+1}{2}}^{|\alpha|} \left|\chi^{(\ell)}\left(\|\xi\|^2 - b\right)\right|,
\end{aligned}$$

because $\operatorname{int}\frac{|\alpha|+1}{2} \le \operatorname{int}\frac{\alpha_1+1}{2} + \operatorname{int}\frac{\alpha_2+1}{2}$. □

LEMMA 6.11. *Let $\chi(t) := e^{1/t}$ for $t < 0$. Then $\chi^{(\ell)}(t) = (-1)^\ell \cdot t^{-2\ell} \cdot Q_\ell(t) \cdot e^{1/t}$ for any $\ell \in \mathbb{N}$, where $Q_\ell \in \mathcal{P}_{\ell-1}$. Moreover, $Q_{\ell+1}(t) = Q_\ell(t) \cdot (1 + 2\ell \cdot t) - t^2 \cdot Q_\ell'(t)$ and therefore, if we put $Q_\ell(t) = \sum_{j=0}^{\ell-1} b_{\ell,j} \cdot t^j$, then*

$$\begin{aligned}
&b_{\ell+1,0} = b_{\ell,0} = 1, \\
&b_{\ell+1,j} = b_{\ell,j} + (2\ell - j + 1) \cdot b_{\ell,j-1} \qquad \text{for all } j \in \mathbb{N} \text{ such that } j < \ell, \\
&b_{\ell+1,\ell} = (\ell+1) \cdot b_{\ell,\ell-1}.
\end{aligned}$$

In particular for $\ell \in \mathbb{N}$ and $j = 0, \dots, \ell - 1$ we have $1 \le b_{\ell,j} \le 2^\ell \cdot (\ell-1)!$ and therefore $|Q_\ell(t)| \le 2^\ell \cdot \ell! \cdot \max\{1, |t|^{\ell-1}\}$ and also $|\chi^{(\ell)}(t)| \le (2\ell)^{2\ell} \cdot e^{-2\ell} \cdot |Q_\ell(t)|$ for all $t < 0$.

Proof. The first part follows by induction. Indeed it is obvious that $\chi'(t) = -t^{-2} \cdot e^{1/t}$, which implies that $Q_1 \equiv 1$ and $b_{1,0} = 1$. For subsequent derivatives we have

$$\chi^{(\ell+1)}(t) = \left(\chi^{(\ell)}\right)'(t) = (-1)^\ell \cdot \left(-2\ell \cdot t^{-2\ell-1} \cdot Q_\ell(t) + t^{-2\ell} \cdot Q_\ell'(t) + t^{-2\ell} \cdot Q_\ell(t) \cdot (-1) \cdot t^{-2}\right) \cdot e^{1/t} =$$
$$= (-1)^{\ell+1} \cdot t^{-2(\ell+1)} \cdot \left(2\ell \cdot t \cdot Q_\ell(t) - t^2 \cdot Q_\ell'(t) + Q_\ell(t)\right) \cdot e^{1/t} = (-1)^{\ell+1} \cdot t^{-2(\ell+1)} \cdot Q_{\ell+1}(t) \cdot e^{1/t}$$

and therefore

$$Q_{\ell+1}(t) = Q_\ell(t) \cdot (1 + 2\ell \cdot t) - t^2 \cdot Q_\ell'(t) =$$
$$= (1 + 2\ell \cdot t) \cdot \sum_{j=0}^{\ell-1} \left(b_{\ell,j} \cdot t^j\right) - t^2 \cdot \sum_{j=0}^{\ell-1} \left(b_{\ell,j} \cdot j \cdot t^{j-1}\right) =$$
$$= \sum_{j=0}^{\ell-1} \left(b_{\ell,j} \cdot t^j\right) + \sum_{j=0}^{\ell-1} \left(2\ell \cdot b_{\ell,j} \cdot t^{j+1}\right) - \sum_{j=0}^{\ell-1} \left(b_{\ell,j} \cdot j \cdot t^{j+1}\right) =$$
$$= \sum_{j=0}^{\ell-1} \left(b_{\ell,j} \cdot t^j\right) + \sum_{j=1}^{\ell} \left(2\ell \cdot b_{\ell,j-1} \cdot t^j\right) - \sum_{j=1}^{\ell} \left(b_{\ell,j-1} \cdot (j-1) \cdot t^j\right) =$$
$$= b_{\ell,0} + \sum_{\substack{j \in \mathbb{N} \\ j < \ell}} \Big(\big(b_{\ell,j} + 2\ell \cdot b_{\ell,j-1} - b_{\ell,j-1} \cdot (j-1)\big) \cdot t^j\Big) + \Big(2\ell \cdot b_{\ell,\ell-1} - b_{\ell,\ell-1} \cdot (\ell-1)\Big) \cdot t^\ell =$$
$$= \sum_{j=0}^{\ell} b_{\ell+1,j} \cdot t^j.$$

It is now obvious that $b_{\ell,j} \ge 1$ for all $\ell \in \mathbb{N}$ and $j = 0, \dots, \ell - 1$.

We put $b_\ell := \max_j b_{\ell,j}$ for all $\ell \in \mathbb{N}$. Then we have $b_1 = b_{1,0} = 1$, $b_{2,0} = 1$, $b_{2,1} = 2b_{1,0} = 2$ and thus $b_2 = 2$. If $\ell \ge 2$ then we have $b_{\ell+1,j} \le 2\ell \cdot b_\ell$ for all $j \in \mathbb{N}$ such that $2 \le j \le \ell$, while $b_{\ell+1,0} = 1$ and $b_{\ell+1,1} = b_{\ell,1} + 2\ell \cdot b_{\ell,0} = b_{\ell,1} + 2\ell$. Altogether by induction we obtain $b_\ell \le 2^\ell \cdot (\ell-1)!$.

Finally, it suffices to note that a function $a(t) := t^{-2\ell} \cdot e^{1/t}$, $t < 0$, attains its maximum at $t_0 = -\frac{1}{2\ell}$.

Proof of proposition 6.8 on cutoff functions. We naturally identify the complex plane $\mathbb{C}$ with $\mathbb{R}^2$ and consider the compact set $\widetilde{K} := \left\{(\xi_1, \xi_2) \in \mathbb{R}^2 \ : \ \xi_1 + \xi_2 \cdot i \in K\right\} \subset\subset \mathbb{R}^2$. Put

$$\chi(t) := \begin{cases} e^{1/t} & \text{for } t < 0, \\ 0 & \text{for } t \ge 0, \end{cases}$$

so that $\chi \in C^\infty(\mathbb{R})$ and let $\Upsilon(\xi) := c_0 \cdot \chi(\|\xi\|^2 - c_1^2)$ for $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$ so that $\Upsilon \in \mathcal{C}^\infty(\mathbb{R}^2)$. Here $c_1 := 3/8$ and $c_0 > 0$ is chosen in such a way that $\int_{\mathbb{R}^2} \Upsilon(\xi)\, d\xi = 1$, where as usual $d\xi = d\xi_1 \cdot d\xi_2$. Put also $\Upsilon_\epsilon(\xi) := \frac{1}{\epsilon^2} \cdot \Upsilon\left(\frac{\xi}{\epsilon}\right)$ for $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$ and note that $\operatorname{supp} \Upsilon_\epsilon = B(0, c_1 \cdot \epsilon) \subset\subset \mathbb{R}^2$.

We define the cutoff function that we are looking for as a convolution of Υ_ϵ with the characteristic function of the set $\widetilde{K}_{\epsilon/2}$

$$\widetilde{u}(\xi) := \int_{\operatorname{dist}(\tau, \widetilde{K}) \le \epsilon/2} \Upsilon_\epsilon(\xi - \tau)\, d\tau$$

and note that

$$0 \le \widetilde{u}(\xi) \le \int_{\mathbb{R}^2} \Upsilon_\epsilon(\xi - \tau)\, d\tau \ = \int_{\mathbb{R}^2} \Upsilon_\epsilon(\tau)\, d\tau \ = \int_{\mathbb{R}^2} \Upsilon(\tau)\, d\tau \ = 1.$$

We also see that $\operatorname{supp} \widetilde{u} = \left\{\xi \in \mathbb{R}^2 : \operatorname{dist}(\xi, \widetilde{K}) \le c_1 \cdot \epsilon + \epsilon/2 = \frac{7}{8}\epsilon\right\}$ and $\widetilde{u}(\xi) = 1$ if $\operatorname{dist}(\xi, \widetilde{K}) \le \frac{\epsilon}{8}$ because in this case

$$\widetilde{u}(\xi) = \int_{\operatorname{dist}(\tau, \widetilde{K}) \le \epsilon/2} \Upsilon_\epsilon(\xi - \tau)\, d\tau \ \ge \int_{B(\xi, c_1 \cdot \epsilon)} \Upsilon_\epsilon(\xi - \tau)\, d\tau \ = \int_{B(0, c_1 \cdot \epsilon)} \Upsilon_\epsilon(\tau)\, d\tau \ = \ 1.$$

Furthermore we have for all $\xi \in \mathbb{R}^2$ and $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}_+^2$ such that $|\alpha| > 0$

$$\begin{aligned}
\left|\frac{\partial^{|\alpha|}}{\partial\xi_1^{\alpha_1}\cdot\partial\xi_2^{\alpha_2}}\widetilde{u}(\xi)\right| &= \left|\int_{\operatorname{dist}(\tau,\widetilde{K})\le\epsilon/2}\frac{\partial^{|\alpha|}}{\partial\xi_1^{\alpha_1}\cdot\partial\xi_2^{\alpha_2}}\Upsilon_\epsilon(\xi-\tau)\,d\tau\right| \le \\
\le \int_{\operatorname{dist}(\tau,\widetilde{K})\le\epsilon/2}\left|\frac{\partial^{|\alpha|}}{\partial\xi_1^{\alpha_1}\cdot\partial\xi_2^{\alpha_2}}\Upsilon_\epsilon(\xi-\tau)\right|\,d\tau &\le \int_{\mathbb{R}^2}\left|\frac{\partial^{|\alpha|}}{\partial\xi_1^{\alpha_1}\cdot\partial\xi_2^{\alpha_2}}\Upsilon_\epsilon(\tau)\right|\,d\tau = \\
= \int_{B(0,c_1\cdot\epsilon)}\left|\frac{\partial^{|\alpha|}}{\partial\xi_1^{\alpha_1}\cdot\partial\xi_2^{\alpha_2}}\Upsilon_\epsilon(\tau)\right|\,d\tau &\le \pi\cdot(c_1\cdot\epsilon)^2\cdot\left\|\frac{\partial^{|\alpha|}}{\partial\xi_1^{\alpha_1}\cdot\partial\xi_2^{\alpha_2}}\Upsilon_\epsilon\right\|_{B(0,c_1\cdot\epsilon)} = \\
= \frac{\pi\cdot(c_1\cdot\epsilon)^2}{\epsilon^{2+|\alpha|}}\cdot\left\|\frac{\partial^{|\alpha|}}{\partial\xi_1^{\alpha_1}\cdot\partial\xi_2^{\alpha_2}}\Upsilon\right\|_{B(0,c_1)} &= \frac{\pi c_1^2}{\epsilon^{|\alpha|}}\cdot\left\|\frac{\partial^{|\alpha|}}{\partial\xi_1^{\alpha_1}\cdot\partial\xi_2^{\alpha_2}}\Upsilon\right\|_{B(0,c_1)}.
\end{aligned}$$

By consecutively applying corollary 6.10, lemma 6.11 and Stirling's formula, according to which $\ell! \le \left(\frac{\ell}{e}\right)^\ell\cdot\sqrt{2\pi\ell}\cdot e^{1/(12\ell)}$, we obtain

$$\begin{aligned}
\left\|\frac{\partial^{|\alpha|}}{\partial\xi_1^{\alpha_1}\cdot\partial\xi_2^{\alpha_2}}\Upsilon\right\|_{B(0,c_1)} &\le c_0\cdot(\alpha_1+1)!\cdot(\alpha_2+1)!\cdot\max\left\{1,c_1^{|\alpha|}\right\}\cdot|\alpha|\cdot\sum_{\ell=\operatorname{int}\frac{|\alpha|+1}{2}}^{|\alpha|}\left\|\chi^{(\ell)}\right\|_{[-c_1^2,0]} \le \\
&\le c_0\cdot(\alpha_1+1)!\cdot(\alpha_2+1)!\cdot|\alpha|\cdot\sum_{\ell=\operatorname{int}\frac{|\alpha|+1}{2}}^{|\alpha|}(2\ell)^{2\ell}\cdot e^{-2\ell}\cdot 2^\ell\cdot\ell! \le \\
&\le c_0\cdot 2\alpha_1^{\alpha_1}\cdot 2\alpha_2^{\alpha_2}\cdot|\alpha|\cdot\sum_{\ell=\operatorname{int}\frac{|\alpha|+1}{2}}^{|\alpha|}\left(\frac{2^2\cdot\ell^2\cdot 2\cdot\ell}{e^2\cdot e}\right)^\ell\cdot\sqrt{2\pi\ell}\cdot e^{1/(12\ell)} \le \\
&\le 4c_0\cdot\sqrt{2\pi}\cdot e^{1/12}\cdot|\alpha|^{\alpha_1}\cdot|\alpha|^{\alpha_2}\cdot|\alpha|\cdot\sum_{\ell=1}^{|\alpha|}\left(\frac{8\ell^3}{e^3}\right)^\ell\cdot\sqrt{\ell} \le \\
\le 11c_0\cdot|\alpha|^{|\alpha|}\cdot|\alpha|\cdot\sum_{\ell=1}^{|\alpha|}\left(\frac{8\cdot|\alpha|^3}{e^3}\right)^{|\alpha|}\cdot\sqrt{|\alpha|} &\le 11c_0\cdot|\alpha|^{|\alpha|}\cdot|\alpha|^{5/2}\cdot\left(\frac{8\cdot|\alpha|^3}{e^3}\right)^{|\alpha|} \le 11c_0\cdot|\alpha|^{4\cdot|\alpha|}
\end{aligned}$$

because $|\alpha|^{5/2} < \left(\frac{5}{2}\right)^{|\alpha|}$ and $\frac{5}{2}\cdot\frac{8}{e^3} < 1$. Hence we conclude that

$$\left\|\frac{\partial^{|\alpha|}}{\partial\xi_1^{\alpha_1}\cdot\partial\xi_2^{\alpha_2}}\,\widetilde{u}\right\|_{\mathbb{R}^2} \le \frac{\pi c_1^2}{\epsilon^{|\alpha|}}\cdot\left\|\frac{\partial^{|\alpha|}}{\partial\xi_1^{\alpha_1}\cdot\partial\xi_2^{\alpha_2}}\Upsilon\right\|_{B(0,c_1)} \le \frac{\pi c_1^2}{\epsilon^{|\alpha|}}\cdot 11c_0\cdot|\alpha|^{4\cdot|\alpha|} \le \frac{d\cdot|\alpha|^{4\cdot|\alpha|}}{\epsilon^{|\alpha|}}$$

where $d := \max\{1, 5c_0\}$ is some absolute constant.

Finally we revert to $K \subset\subset \mathbb{C}$ for which we define the cutoff function $u \in \mathcal{C}^\infty(\mathbb{C})$ as follows:

$$u(z) := \widetilde{u}\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right).$$

Properties (a), (b) and (c) are obvious, while property (d) can easily be proved by mathematical induction. Indeed if we write

$$D^\alpha u(z) = \sum_{\substack{\beta\in\mathbb{Z}_+^2\\|\beta|=|\alpha|}} c_{\alpha,\beta}\cdot\frac{\partial^{|\beta|}}{\partial\xi_1^{\beta_1}\cdot\partial\xi_2^{\beta_2}}\,\widetilde{u}\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right)$$

for some $c_{\alpha,\beta} \in \mathbb{C}$, with $\alpha, \beta \in \mathbb{Z}_+^2$, then we have

$$\begin{aligned}
D^{(\alpha_1+1,\alpha_2)}u(z) &= \sum_{|\beta|=|\alpha|} c_{\alpha,\beta}\cdot\left(\frac{1}{2}\cdot\frac{\partial^{|\beta|+1}}{\partial\xi_1^{\beta_1+1}\cdot\partial\xi_2^{\beta_2}}\,\widetilde{u} + \frac{1}{2i}\cdot\frac{\partial^{|\beta|+1}}{\partial\xi_1^{\beta_1}\cdot\partial\xi_2^{\beta_2+1}}\,\widetilde{u}\right)\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right), \\
D^{(\alpha_1,\alpha_2+1)}u(z) &= \sum_{|\beta|=|\alpha|} c_{\alpha,\beta}\cdot\left(\frac{1}{2}\cdot\frac{\partial^{|\beta|+1}}{\partial\xi_1^{\beta_1+1}\cdot\partial\xi_2^{\beta_2}}\,\widetilde{u} - \frac{1}{2i}\cdot\frac{\partial^{|\beta|+1}}{\partial\xi_1^{\beta_1}\cdot\partial\xi_2^{\beta_2+1}}\,\widetilde{u}\right)\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right).
\end{aligned}$$

Therefore we see that

$$\sum_{|\beta|=|\alpha|+1} |c_{(\alpha_1+1,\alpha_2),\beta}| \leq \sum_{|\beta|=|\alpha|} |c_{\alpha,\beta}|,$$
$$\sum_{|\beta|=|\alpha|+1} |c_{(\alpha_1,\alpha_2+1),\beta}| \leq \sum_{|\beta|=|\alpha|} |c_{\alpha,\beta}|,$$

which implies that for all $\alpha \in \mathbb{Z}_+^2$

$$\sum_{|\beta|=|\alpha|} |c_{\alpha,\beta}| \leq |c_{(0,0),(0,0)}| = 1,$$

$$\|D^\alpha u(z)\|_{\mathbb{C}} \leq \sum_{|\beta|=|\alpha|} |c_{\alpha,\beta}| \cdot \left\| \frac{\partial^{|\beta|}}{\partial \xi_1^{\beta_1} \cdot \partial \xi_2^{\beta_2}} \widetilde{u} \right\|_{\mathbb{R}^2} \leq \frac{d \cdot |\alpha|^{4\cdot|\alpha|}}{\epsilon^{|\alpha|}}. \quad \square$$

Proposition 6.12. *For any compact set $E \subset\subset \mathbb{C}$, $0 < \delta \leq 1$, $f \in \mathcal{H}^\infty(E_\delta)$ and $\ell \in \mathbb{N}$ we have*

$$\langle\!\langle f \rangle\!\rangle_{E_{\delta/17},\ell} \leq \frac{(2d \cdot \ell)^{5\ell}}{\delta^\ell} \cdot ||f||_{E_\delta}$$

where $d \geq 1$ is the absolute constant from proposition 6.8 on cutoff functions.

Proof. Let $u \in \mathcal{C}^\infty(\mathbb{C})$ be the cutoff function constructed in proposition 6.8 for the compact set $K := E$ and radius $\epsilon := \delta/2$. We put $\widetilde{f} := u \cdot f$ and see that $\widetilde{f} \in \mathcal{H}^\infty(E_{\delta/17})$ because $\widetilde{f} \equiv f$ on $E_{\delta/16}$, which contains an open neighbourhood of the set $E_{\delta/17}$. In $\mathbb{C} \setminus E_{\delta/2}$ we have $u \equiv 0$ and hence $D^\alpha \widetilde{f} \equiv 0$ for all $\alpha \in \mathbb{Z}_+^2$. Consequently by the definition of the holomorphic quotient norms we have

$$\langle\!\langle f \rangle\!\rangle_{E_{\delta/17},\ell} \leq \|\widetilde{f}\|_{\operatorname{conv} E_{\delta/17},\ell} \leq \|\widetilde{f}\|_{\mathbb{C},\ell} = \|\widetilde{f}\|_{E_{\delta/2},\ell}.$$

By the Leibniz rule we obtain for every $z \in E_{\delta/2}$ and $\alpha \in \mathbb{Z}_+^2$ such that $|\alpha| = \ell$

$$D^\alpha \widetilde{f}(z) = \sum_{\substack{\beta \in \mathbb{Z}_+^2 \\ \beta \leq \alpha}} \binom{\alpha}{\beta} \cdot D^\beta u(z) \cdot D^{\alpha-\beta} f(z) =$$

$$= \sum_{\substack{\beta \leq \alpha \\ \beta_2 = \alpha_2}} \binom{\alpha}{\beta} \cdot D^\beta u(z) \cdot D^{\alpha-\beta} f(z) = \sum_{\beta_1 \leq \alpha_1} \binom{\alpha_1}{\beta_1} \cdot D^{(\beta_1,\alpha_2)} u(z) \cdot f^{(\alpha_1-\beta_1)}(z)$$

because f is holomorphic in E_δ. Note that the expression $\beta \leq \alpha$ means that $\beta_1 \leq \alpha_1$ and $\beta_2 \leq \alpha_2$, while $\binom{\alpha}{\beta} := \binom{\alpha_1}{\beta_1} \cdot \binom{\alpha_2}{\beta_2}$. The properties of the cutoff function and Cauchy's integral formula lead us to

$$|D^\alpha \widetilde{f}(z)| \leq \sum_{\beta_1 \leq \alpha_1} \binom{\alpha_1}{\beta_1} \cdot |D^{(\beta_1,\alpha_2)} u(z)| \cdot |f^{(\alpha_1-\beta_1)}(z)| \leq \sum_{\beta_1 \leq \alpha_1} \binom{\alpha_1}{\beta_1} \cdot \frac{C_{\beta_1+\alpha_2}}{(\delta/2)^{\beta_1+\alpha_2}} \cdot ||f^{(\alpha_1-\beta_1)}||_{E_{\delta/2}} \leq$$
$$\leq \sum_{\beta_1 \leq \alpha_1} \binom{\alpha_1}{\beta_1} \cdot \frac{C_{\beta_1+\alpha_2}}{(\delta/2)^{\beta_1+\alpha_2}} \cdot \frac{(\alpha_1-\beta_1)!}{(\delta/2)^{\alpha_1-\beta_1}} \cdot ||f||_{E_\delta} = \sum_{\beta_1 \leq \alpha_1} \frac{\alpha_1!}{\beta_1!} \cdot \frac{C_{\beta_1+\alpha_2}}{(\delta/2)^{|\alpha|}} \cdot ||f||_{E_\delta} \leq$$
$$\leq \sum_{\beta_1 \in \mathbb{Z}_+} \frac{|\alpha|!}{\beta_1!} \cdot \frac{C_{|\alpha|}}{(\delta/2)^{|\alpha|}} \cdot ||f||_{E_\delta} = \frac{e \cdot \ell! \cdot C_\ell}{(\delta/2)^\ell} \cdot ||f||_{E_\delta} \leq \frac{2e \cdot \ell^\ell \cdot C_\ell}{\delta^\ell} \cdot ||f||_{E_\delta} = \frac{2e \cdot d \cdot \ell^{5\ell}}{\delta^\ell} \cdot ||f||_{E_\delta},$$

because $2^\ell \cdot \ell! \leq 2\ell^\ell$. Finally we see that

$$\langle\!\langle f \rangle\!\rangle_{E_{\delta/17},\ell} \leq \|\widetilde{f}\|_{E_{\delta/2},\ell} = \|\widetilde{f}\|_{E_{\delta/2}} + \sum_{|\alpha|=\ell} \|D^\alpha \widetilde{f}\|_{E_{\delta/2}} \leq$$
$$\leq (\ell+2) \cdot \frac{2e \cdot d \cdot \ell^{5\ell}}{\delta^\ell} \cdot ||f||_{E_\delta} \leq \frac{(2d \cdot \ell)^{5\ell}}{\delta^\ell} \cdot ||f||_{E_\delta},$$

because $(\ell+2) \cdot 2e \leq 2^{5\ell}$. $\square$

CHAPTER VII

SOBOLEV PROPERTY FOR HOLOMORPHIC FUNCTIONS (SPH)

DEFINITION 7.1. A compact set $E \subset\subset \mathbb{C}$ admits the Sobolev Property for Holomorphic functions SPH(m,s,k) where $m,s,k \geq 1$, if

$$\exists c_0 \geq 0 \quad \forall \ell \in \mathbb{N} \quad \exists c_\ell \geq 1 \quad \forall j \in \mathbb{N} \text{ such that } \ell \geq m\cdot j \quad \forall 0<\delta\leq 1 \quad \forall f \in \mathcal{H}^\infty(E_\delta) \quad : \\ |f|_{E,j} \leq \left(\frac{c_\ell}{\delta^s}\right)^{j+c_0} \cdot ||f||_E^{1-\frac{m\cdot j}{\ell}} \cdot ||f||_{E_\delta}^{\frac{m\cdot j}{\ell}}$$

and additionally $c_\ell \leq c_1 \cdot \ell^k$. Without the last assumption we speak of the Weak Sobolev Property for Holomorphic functions WSPH(m,s). We will write that the set E admits SPH, respectively WSPH, if it admits SPH(m,s,k), respectively WSPH(m,s), for some $m,s,k \geq 1$.

THEOREM 7.2. *For any compact set $E \subset\subset \mathbb{C}$, $m,k \geq 1$ and $s \geq 0$ we have*

$$\mathrm{SPQH}(m,s,k) \implies \mathrm{SPH}\big(m, m+s, k+5m\big), \\ \mathrm{WSPQH}(m,s) \implies \mathrm{WSPH}(m, m+s).$$

PROOF. Let's first assume that the set E admits WSPQH(m,s), i.e.

$$\forall \ell \in \mathbb{N} \quad \exists c_\ell \geq 1 \quad \forall j \in \mathbb{N} \text{ such that } \ell \geq m\cdot j \quad \forall 0<\delta\leq 1 \quad \forall f \in \mathcal{H}^\infty(E_\delta) \quad : \\ |f|_{E,j} \leq \left(\frac{c_\ell}{\delta^s}\right)^{j} \cdot \langle\!\langle f \rangle\!\rangle_E^{1-\frac{m\cdot j}{\ell}} \cdot \langle\!\langle f \rangle\!\rangle_{E_\delta,\ell}^{\frac{m\cdot j}{\ell}}.$$

Fix $\ell \in \mathbb{N}$, $j \in \mathbb{N}$ such that $\ell \geq m\cdot j$, $0<\delta\leq 1$ and $f \in \mathcal{H}^\infty(E_\delta)$. We combine WSPQH with proposition 6.12 to obtain

$$\begin{aligned} |f|_{E,j} &\leq \left(\frac{c_\ell}{(\delta/17)^s}\right)^j \cdot \langle\!\langle f \rangle\!\rangle_E^{1-\frac{m\cdot j}{\ell}} \cdot \langle\!\langle f \rangle\!\rangle_{E_{\delta/17},\ell}^{\frac{m\cdot j}{\ell}} \leq \left(\frac{17^s\cdot c_\ell}{\delta^s}\right)^j \cdot ||f||_E^{1-\frac{m\cdot j}{\ell}} \cdot \frac{(2d\cdot\ell)^{5m\cdot j}}{\delta^{m\cdot j}} \cdot ||f||_{E_\delta}^{\frac{m\cdot j}{\ell}} = \\ &= \left(\frac{17^s\cdot c_\ell \cdot (2d\cdot \ell)^{5m}}{\delta^{m+s}}\right)^j \cdot ||f||_E^{1-\frac{m\cdot j}{\ell}} \cdot ||f||_{E_\delta}^{\frac{m\cdot j}{\ell}} = \left(\frac{\widetilde{c}_\ell}{\delta^{m+s}}\right)^j \cdot ||f||_E^{1-\frac{m\cdot j}{\ell}} \cdot ||f||_{E_\delta}^{\frac{m\cdot j}{\ell}}, \end{aligned}$$

where $\widetilde{c}_\ell := 17^s \cdot c_\ell \cdot (2d\cdot\ell)^{5m}$ and $d \geq 1$ depends solely on the choice of the set E. This proves that E admits WSPH$(m, m+s)$ but also if $c_\ell \leq c_1 \cdot \ell^k$ then $\widetilde{c}_\ell \leq 17^s \cdot c_1 \cdot \ell^k \cdot (2d\cdot\ell)^{5m} = \widetilde{c}_1 \cdot \ell^{k+5m}$. □

Now in order to prove the implication SPH $\implies$ LMP, respectively WSPH $\implies$ WLMP, we need to do some preparations.

LEMMA 7.3 [Białas-Eggink 2, lemma 2.1]. *Let p be a polynomial of degree n and $z_0,\ldots,z_n$ be arbitrary points of $\mathbb{C}$ such that $z_\mu \neq z_\nu$ as $\mu \neq \nu$. Then for each $j = 1,\ldots,n$ we have*

$$\frac{1}{j}\cdot p^{(j)}(z_0) = \sum_{\mu=1}^{n} \left(\frac{p(z_\mu)-p(z_0)}{z_\mu - z_0} \cdot \left[\frac{d^{j-1}}{dz^{j-1}} \prod_{\substack{\nu=1,\ldots,n \\ \nu\neq\mu}} \frac{z-z_\nu}{z_\mu - z_\nu} \right]_{/z=z_0} \right).$$

In particular,

$$p'(z_0) = \sum_{\mu=1}^{n} \left(\frac{p(z_\mu)-p(z_0)}{z_\mu - z_0} \cdot \prod_{\substack{\nu=1,\ldots,n \\ \nu\neq\mu}} \frac{z_0-z_\nu}{z_\mu - z_\nu} \right).$$

Proof. Put $q(z) := \frac{p(z)-p(z_0)}{z-z_0}$. It is evident that q is a polynomial of degree $n-1$ and

$$q(z) = \sum_{j=1}^{n} \frac{1}{j!} \cdot p^{(j)}(z_0) \cdot (z-z_0)^{j-1}.$$

Therefore

$$q^{(j-1)}(z_0) = \frac{1}{j} \cdot p^{(j)}(z_0) \quad \text{for } j = 1, \ldots, n$$

and by the Lagrange interpolation formula we have for arbitrary $z \in \mathbb{C}$

$$q(z) = \sum_{\mu=1}^{n} \left(q(z_\mu) \cdot \prod_{\substack{\nu=1,\ldots,n \\ \nu \neq \mu}} \frac{z-z_\nu}{z_\mu - z_\nu} \right). \quad \square$$

Proposition 7.4 [Białas-Eggink 2, proposition 2.2]. *Let $E \subset\subset \mathbb{C}$, $z_0 \in E$, $r > 0$ and $n \in \mathbb{N}$ be fixed. Put*

$$T = T(z_0, r) := \{t \in [0, r] \ : \ \exists z \in E \text{ such that } |z - z_0| = t\}.$$

If there exists a constant $c_n > 0$ such that for every polynomial $q \in \mathcal{P}_n(\mathbb{R})$ we have

$$|q'(0)| \leq c_n \cdot ||q||_T, \tag{1}$$

then for every polynomial $p \in \mathcal{P}_n(\mathbb{C})$ it follows that

$$|p'(z_0)| \leq 2n \cdot c_n \cdot ||p||_{E \cap B(z_0, r)}. \tag{2}$$

Proof. Consider $n+1$ Fekete extremal points $t_0, \ldots, t_n$ of the set T constructed as follows. Put

$$V(x_0, \ldots, x_n) := \prod_{0 \leq \mu < \nu \leq n} (x_\nu - x_\mu).$$

We choose $t_0, \ldots, t_n \in T$ such that

$$|V(t_0, \ldots, t_n)| = \max \{|V(x_0, \ldots, x_n)| \ : \ x_0, \ldots, x_n \in T\}. \tag{3}$$

We can assume that t_0 is the smallest number of $t_0, \ldots, t_n$. Observe that $t_0 = 0$. Indeed, $t_\nu - t_0 \leq t_\nu$ for all $\nu = 1, \ldots, n$ and from condition (3) we deduce that $t_0 = 0$.

By inequality (1), the set T contains at least $n+1$ points, and consequently $V(t_0, \ldots, t_n) \neq 0$. For $\mu = 0, 1, \ldots, n$ consider the Lagrange polynomials

$$L_\mu(t) := \frac{V(t_0, \ldots, t_{\mu-1}, t, t_{\mu+1}, \ldots, t_n)}{V(t_0, \ldots, t_n)} = \prod_{\substack{\nu=0,\ldots,n \\ \nu \neq \mu}} \frac{t - t_\nu}{t_\mu - t_\nu}. \tag{4}$$

We have $||L_\mu||_T = 1$, as is easy to check. By inequality (1), for $\mu = 1, \ldots, n$ we have

$$\frac{\prod\limits_{\substack{\nu=1,\ldots,n \\ \nu \neq \mu}} t_\nu}{\prod\limits_{\substack{\nu=0,\ldots,n \\ \nu \neq \mu}} |t_\mu - t_\nu|} = |L'_\mu(0)| \leq c_n \cdot ||L_\mu||_T = c_n.$$

Now choose $z_1, \ldots, z_n \in E \cap B(z_0, r)$ such that $|z_\nu - z_0| = t_\nu$ for $\nu = 1, \ldots, n$ and fix $p \in \mathcal{P}_n$. Lemma 7.3 implies that

$$|p'(z_0)| \; = \; \left| \sum_{\mu=1}^{n} \left(\frac{p(z_\mu)-p(z_0)}{z_\mu - z_0} \cdot \prod_{\substack{\nu=1,\dots,n \\ \nu\neq\mu}} \frac{z_0 - z_\nu}{z_\mu - z_\nu} \right) \right| \; \leq \; 2 \cdot ||p||_{E\cap B(z_0,r)} \cdot \sum_{\mu=1}^{n} \left(\frac{\prod\limits_{\substack{\nu=1,\dots,n \\ \nu\neq\mu}} |z_0 - z_\nu|}{\prod\limits_{\substack{\nu=0,\dots,n \\ \nu\neq\mu}} |z_\mu - z_\nu|} \right).$$

It is easily seen that $|z_\mu - z_\nu| \geq |t_\mu - t_\nu|$ for each μ and ν. By the above, we obtain

$$|p'(z_0)| \; \leq \; 2 \cdot ||p||_{E\cap B(z_0,r)} \cdot \sum_{\mu=1}^{n} \left(\frac{\prod\limits_{\substack{\nu=1,\dots,n \\ \nu\neq\mu}} t_\nu}{\prod\limits_{\substack{\nu=0,\dots,n \\ \nu\neq\mu}} |t_\mu - t_\nu|} \right) \; =$$

$$= 2 \cdot ||p||_{E\cap B(z_0,r)} \cdot \sum_{\mu=1}^{n} |L'_\mu(0)| \leq 2n \cdot c_n \cdot ||p||_{E\cap B(z_0,r)}. \quad \square$$

Remark 7.5 [Białas-Eggink 2, remark 2.4]. Note that inequality (2) does not imply inequality (1). It is sufficient to consider the set $E := \{0\} \cup \{z \in \mathbb{C} \; : \; |z| = r\}$ and $z_0 = 0$. By Cauchy's integral formula, inequality (2) is satisfied for all polynomials $p \in \mathcal{P}_n$, but the set $T = \{0, r\}$ does not admit any Markov inequality.

Note that in the proof of proposition 7.4 we did not need inequality (1) for all polynomials but only for those of Lagrange.

Corollary 7.6 [Białas-Eggink 2, corollary 2.5]. *In proposition 7.4, it is sufficient to assume that*

$$|L'_\mu(0)| \leq c_n \cdot ||L_\mu||_{T(z_0,r)} \quad \text{for } \mu = 1, \dots, n.$$

Corollary 7.7. *If a compact set $E \subset\subset \mathbb{C}$ is connected, then it admits $LMP(1,3)$.*

Proof. Without loss of generality we can assume that $\operatorname{diam} E \geq 2$. Now note that for each $z_0 \in E$ and $0 < r \leq 1$ the set $T = T(z_0, r)$ as defined in proposition 7.4 is connected and it contains the points 0 and r. Therefore $T = [0, r]$ and it admits the classic Markov inequality stated in theorem 1.1, i.e.

$$|q'(0)| \leq \frac{2n^2}{r} \cdot ||q||_T$$

for every polynomial $q \in \mathcal{P}_n(\mathbb{R})$. Consequently by proposition 7.4 we have

$$|p'(z_0)| \leq \frac{4n^3}{r} \cdot ||p||_{E\cap B(z_0,r)}$$

for every polynomial $p \in \mathcal{P}_n(\mathbb{C})$. $\square$

The next lemma was inspired by [Zeriahi, theorem 2.1].

Lemma 7.8 [Białas-Eggink 2, lemma 2.7]. *If $p \in \mathcal{P}_n$ and $r > 0$, then there exists an interval $I \subset [0, r]$ of length at least $\frac{r}{4n^2}$ such that*

$$||p||_{[0,r]} \leq 2 \cdot |p(x)| \quad \text{for all } x \in I.$$

Proof. Let x_0 be a point of $[0, r]$ such that $|p(x_0)| = ||p||_{[0,r]}$. Put $I := \left[x_0 - \frac{r}{4n^2}, x_0 + \frac{r}{4n^2}\right] \cap \left[0, r\right]$ and consider an arbitrary point $x \in I$. The mean value theorem leads to

$$|p(x_0) - p(x)| \leq ||p'||_I \cdot |x_0 - x| \leq \frac{r}{4n^2} \cdot ||p'||_{[0,r]}.$$

The interval $[0, r]$ admits the classic Markov inequality, hence

$$||p'||_{[0,r]} \leq \frac{2n^2}{r} \cdot ||p||_{[0,r]} = \frac{2n^2}{r} \cdot |p(x_0)|.$$

From the above it follows that

$$|p(x_0)| - |p(x)| \leq |p(x_0) - p(x)| \leq \frac{r}{4n^2} \cdot \frac{2n^2}{r} \cdot |p(x_0)| = \frac{1}{2} \cdot |p(x_0)|,$$

and finally we have

$$||p||_{[0,r]} = |p(x_0)| \leq 2 \cdot |p(x)|. \quad \square$$

THEOREM 7.9 [cf. Białas-Eggink 2, theorem 2.8; cf. Bos-Milman, theorem A]. *For any compact set* $E \subset\subset \mathbb{C}$ *and* $m, s, k \geq 1$ *we have*

$$\begin{aligned} \mathrm{SPH}(m,s,k) &\implies \mathrm{LMP}(m',k'), \\ \mathrm{WSPH}(m,s) &\implies \mathrm{WLMP}(m'). \end{aligned}$$

for any $m' > s$ *and* $k' > k + 3s$.

PROOF. Let's first assume that the set E admits $\mathrm{WSPH}(m,s)$, i.e.

$$\exists c_0 \geq 0 \quad \forall \ell \in \mathbb{N} \quad \exists c_\ell \geq 1 \quad \forall j \in \mathbb{N} \text{ such that } \ell \geq m \cdot j \quad \forall 0 < \delta \leq 1 \quad \forall f \in \mathcal{H}^\infty(E_\delta) \quad : \\ |f|_{E,j} \leq \left(\frac{c_\ell}{\delta^s}\right)^{j+c_0} \cdot ||f||_E^{1-\frac{m\cdot j}{\ell}} \cdot ||f||_{E_\delta}^{\frac{m\cdot j}{\ell}}.$$

Without loss of generality we can assume that the sequence $\{c_\ell\}_{\ell\in\mathbb{N}}$ is increasing. Fix an arbitrary integer $a \geq m+1$ and put $m_a := \frac{s\cdot(a+c_0)-m}{a-m} \geq 1$ and $k_a := \frac{(k+2s)\cdot(a+c_0)}{a-m} > 3$. We will use proposition 3.3 to prove $\mathrm{LMP}(m_a, k_a+m_a)$ respectively $\mathrm{WLMP}(m_a)$, which for $a \to \infty$ leads to the assertion of the theorem, because $\lim_{a\to\infty} m_a = s$ and $\lim_{a\to\infty}(k_a+m_a) = k+3s$.

Hence we need to prove the assumption of proposition 3.3, i.e.

$$\forall n \in \mathbb{N} \quad \exists \widetilde{c}_{a,n} \geq 1 \quad \forall z_0 \in E \quad \forall 0 < r \leq 1 \quad \forall p \in \mathcal{P}_n \quad : \quad |p'(z_0)| \leq \frac{\widetilde{c}_{a,n}}{r^{m_a}} \cdot \|p\|_{E\cap B(z_0,r)} \tag{5}$$

and appropriately control the coefficients $\widetilde{c}_{a,n}$ if required. For $n \in \mathbb{N}$ we put

$$\widetilde{c}_{a,n} := \max\left\{\left(4\cdot 3^s \cdot c_{2a^2\cdot n}\right)^{\frac{a+c_0}{a-m}} \cdot \left(4n^2\right)^{\frac{s\cdot(a+c_0)}{a-m}}, 8n^3\right\}.$$

Note that if $c_\ell \leq c_1 \cdot \ell^k$ then

$$\begin{aligned} \widetilde{c}_{a,n} &\leq \max\left\{\left(4\cdot 3^s \cdot c_1 \cdot (2a^2\cdot n)^k\right)^{\frac{a+c_0}{a-m}} \cdot \left(4n^2\right)^{\frac{s\cdot(a+c_0)}{a-m}}, 8n^3\right\} = \\ &= \left(4\cdot 3^s \cdot c_1 \cdot (2a^2)^k\right)^{\frac{a+c_0}{a-m}} \cdot 4^{\frac{s\cdot(a+c_0)}{a-m}} \cdot n^{k_a}, \end{aligned}$$

and therefore these coefficients can be controlled as required.

We proceed to prove inequality (5). Fix arbitrarily $n \in \mathbb{N}$, $z_0 \in E$ and $0 < r \leq 1$. Define $T = T(z_0, r)$ as in proposition 7.4. Choose $t_0, \ldots, t_n \in T$ satisfying condition (3). As in the proof of proposition 7.4, we can assume that $t_0 = 0$. For $\mu = 1, \ldots, n$ denote by L_μ the Lagrange polynomial given by definition (4). Let I_μ be an interval of length at least $\frac{r}{4n^2}$ constructed for the polynomial L_μ as in lemma 7.8.

If for every $\mu = 1, \ldots, n$ there exists $z_\mu \in E$ such that $|z_\mu - z_0| \in I_\mu$, then we use proposition 7.4. Specifically, in this case for $\mu = 1, \ldots, n$, I_μ meets $T(z_0, r)$, say at t_μ. By the classic Markov inequality for the interval $[0, r]$ and applying lemma 7.8 we obtain for $\mu = 1, \ldots, n$

$$|L'_\mu(0)| \leq \frac{2n^2}{r} \cdot ||L_\mu||_{[0,r]} \leq \frac{4n^2}{r} \cdot |L_\mu(t_\mu)| \leq \frac{4n^2}{r} \cdot ||L_\mu||_{T(z_0,r)}.$$

Hence by proposition 7.4 and corollary 7.6 we have

$$|p'(z_0)| \leq \frac{8n^3}{r} \cdot ||p||_{E\cap B(z_0,r)} \leq \frac{\widetilde{c}_{a,n}}{r^{m_a}} \cdot \|p\|_{E\cap B(z_0,r)}$$

for all polynomials $p \in \mathcal{P}_n$ as required in the assumption of proposition 3.3.

We now turn to the case where $I_\mu \cap T(z_0, r) = \emptyset$ for some $\mu \in \{1, \ldots, n\}$, which implies that there is an empty annulus around z_0 of a certain minimum size. We shall have established the theorem if we prove that in this case we have for all $p \in \mathcal{P}_n$

$$|p'(z_0)| \leq (2b_{a,n})^{\frac{a+c_0}{a-m}} \cdot \left(\frac{r^m}{l_\mu^{s\cdot(a+c_0)}}\right)^{\frac{1}{a-m}} \cdot ||p||_{E\cap B(z_0,r)}, \tag{6}$$

where

$$b_{a,n} := 2 \cdot 3^s \cdot c_{2a^2 \cdot n}$$

and l_μ is the length of I_μ. Indeed, by lemma 7.8, $l_\mu \geq \frac{r}{4n^2}$ and applying this to inequality (6) we obtain

$$|p'(z_0)| \leq (2b_{a,n})^{\frac{a+c_0}{a-m}} \cdot \left(\frac{(4n^2)^{s\cdot(a+c_0)}}{r^{s\cdot(a+c_0)-m}} \right)^{\frac{1}{a-m}} \cdot ||p||_{E\cap B(z_0,r)} \leq \frac{\widetilde{c}_{a,n}}{r^{m_a}} \cdot ||p||_{E\cap B(z_0,r)},$$

as required.

It remains to show inequality (6) for all $p \in \mathcal{P}_n$. For this purpose, we write $[\rho_0, \rho_1]$ for the interval I_μ. Of course, $0 < \rho_0 < \rho_1 \leq r$ and $\rho_1 - \rho_0 = l_\mu$.

The rest of the proof is adapted from [Bos-Milman, theorem A].

Let $\varepsilon \in \mathcal{C}^\infty(\mathbb{R})$ be any cutoff function with the following properties:

$$\begin{aligned} &(a) \quad 0 \leq \varepsilon(x) \leq 1 && \text{for all } x \in \mathbb{R}, \\ &(b) \quad \varepsilon(x) = 1 && \text{for } x \leq \frac{1}{3}, \\ &(c) \quad \varepsilon(x) = 0 && \text{for } x \geq \frac{2}{3}, \end{aligned}$$

and put $h(z) := \varepsilon\left(\frac{|z-z_0|-\rho_0}{l_\mu}\right)$ so that $h \in \mathcal{C}^\infty(\mathbb{C})$.

Now fix arbitrarily $p \in \mathcal{P}_n$ and let $q(z) := \big(p(z) - p(z_0)\big)^a$, $q \in \mathcal{P}_{a\cdot n}$, and $f(z) := h(z) \cdot q(z)$. Choose any $\ell \in \mathbb{N}$ such that $a^2 \cdot n \leq \ell \leq 2a^2 \cdot n$. Since $T(z_0, r) \cap [\rho_0, \rho_0 + l_\mu] = \emptyset$, we have $f \in \mathcal{H}^\infty(E_\delta)$, where $\delta := \frac{l_\mu}{3} \leq \frac{r}{3}$.

We see that

$$\|f\|_E = \|f\|_{E\cap B(z_0,\rho_1)} \leq \|q\|_{E\cap B(z_0,\rho_1)}, \tag{7}$$

$$\|f\|_{E_\delta} = \|f\|_{E_\delta \cap B(z_0,\rho_1)} \leq \|f\|_{B(z_0,\rho_1)} \leq \|q\|_{B(z_0,\rho_1)}, \tag{8}$$

and by the WSPH(m, s) we have for arbitrary $j \in \mathbb{N}$ such that $j \leq a \cdot n \leq \frac{\ell}{a} < \frac{\ell}{m}$

$$|q^{(j)}(z_0)| = \left| \frac{\partial^j f}{\partial z^j}(z_0) \right| \leq \left|\left| \frac{\partial^j f}{\partial z^j} \right|\right|_E = |f|_{E,j} \leq \left(\frac{c_\ell}{\delta^s} \right)^{j+c_0} \cdot ||f||_E^{1-\frac{m\cdot j}{\ell}} \cdot ||f||_{E_\delta}^{\frac{m\cdot j}{\ell}}. \tag{9}$$

From inequalities (7), (8), (9) and the choice of ℓ and δ it follows that

$$\begin{aligned} |q^{(j)}(z_0)| &\leq \left(\frac{3^s \cdot c_\ell}{l_\mu^s} \right)^{j+c_0} \cdot ||q||_{E\cap B(z_0,\rho_1)}^{1-\frac{m\cdot j}{\ell}} \cdot ||q||_{B(z_0,\rho_1)}^{\frac{m\cdot j}{\ell}} \leq \\ &\leq \left(\frac{\frac{1}{2} b_{a,n}}{l_\mu^s} \right)^{j+c_0} \cdot ||q||_{E\cap B(z_0,r)}^{1-\frac{m\cdot j}{\ell}} \cdot ||q||_{B(z_0,r)}^{\frac{m\cdot j}{\ell}}. \end{aligned} \tag{10}$$

Our next objective is to estimate

$$\lambda := \left(\frac{||q||_{B(z_0,r)}}{||q||_{E\cap B(z_0,r)}} \right)^{\frac{1}{a\cdot n}} \geq 1.$$

By inequality (10), we have for any $j = 1, \ldots, a \cdot n$

$$|q^{(j)}(z_0)| \leq \left(\frac{\frac{1}{2} b_{a,n}}{l_\mu^s} \right)^{j+c_0} \cdot \lambda^{\frac{a\cdot n\cdot m\cdot j}{\ell}} \cdot ||q||_{E\cap B(z_0,r)} \leq \left(\frac{\frac{1}{2} b_{a,n}}{l_\mu^s} \right)^{j+c_0} \cdot \lambda^{\frac{m\cdot j}{a}} \cdot ||q||_{E\cap B(z_0,r)} \tag{11}$$

and for $j = 0$ this is trivially true too. From this, applying Taylor's formula and the fact that $l_\mu \le r \le 1$, we get

$$||q||_{B(z_0,r)} \le \sum_{j=0}^{a\cdot n} \frac{r^j}{j!} \cdot |q^{(j)}(z_0)| \le \sum_{j=0}^{a\cdot n} \left(\frac{1}{2} b_{a,n}\right)^{j+c_0} \cdot \left(\frac{r}{l_\mu^s}\right)^j \cdot \frac{1}{l_\mu^{s\cdot c_0}} \cdot \lambda^{\frac{m\cdot j}{a}} \cdot ||q||_{E\cap B(z_0,r)} \le$$
$$\le 2 \cdot (b_{a,n})^{a\cdot n + c_0} \cdot \left(\frac{r}{l_\mu^s}\right)^{a\cdot n} \cdot \frac{1}{l_\mu^{s\cdot c_0}} \cdot \lambda^{m\cdot n} \cdot ||q||_{E\cap B(z_0,r)}.$$

It follows that

$$\lambda^{a\cdot n} = \frac{||q||_{B(z_0,r)}}{||q||_{E\cap B(z_0,r)}} \le 2 \cdot (b_{a,n})^{a\cdot n + c_0} \cdot \left(\frac{r}{l_\mu^s}\right)^{a\cdot n} \cdot \frac{1}{l_\mu^{s\cdot c_0}} \cdot \lambda^{m\cdot n},$$

and consequently

$$\lambda \le 2 \cdot (b_{a,n})^{\frac{a + c_0/n}{a-m}} \cdot \left(\frac{r}{l_\mu^s}\right)^{\frac{a}{a-m}} \cdot \left(\frac{1}{l_\mu^{s\cdot c_0}}\right)^{\frac{1}{(a-m)\cdot n}}.$$

By combining this estimate with inequality (11) we can assert that for $j = 0, \dots, a \cdot n$

$$|q^{(j)}(z_0)| \le \left(\frac{\frac{1}{2} b_{a,n}}{l_\mu^s}\right)^{j+c_0} \cdot 2^{\frac{m\cdot j}{a}} \cdot (b_{a,n})^{\frac{m\cdot j\cdot(1+c_0/(a\cdot n))}{a-m}} \cdot \left(\frac{r}{l_\mu^s}\right)^{\frac{m\cdot j}{a-m}} \cdot \left(\frac{1}{l_\mu^{s\cdot c_0}}\right)^{\frac{m\cdot j}{a\cdot(a-m)\cdot n}} \cdot ||q||_{E\cap B(z_0,r)} \le$$
$$\le \left(\frac{b_{a,n}}{l_\mu^s}\right)^{j+c_0} \cdot (b_{a,n})^{\frac{m\cdot(j+c_0)}{a-m}} \cdot \left(\frac{r}{l_\mu^s}\right)^{\frac{m\cdot j}{a-m}} \cdot \left(\frac{1}{l_\mu^{s\cdot c_0}}\right)^{\frac{m}{a-m}} \cdot ||q||_{E\cap B(z_0,r)} \le$$
$$\le (b_{a,n})^{\frac{a\cdot(j+c_0)}{a-m}} \cdot \left(\frac{r^m}{l_\mu^{s\cdot a}}\right)^{\frac{j}{a-m}} \cdot \left(\frac{1}{l_\mu^{s\cdot c_0}}\right)^{1+\frac{m}{a-m}} \cdot ||q||_{E\cap B(z_0,r)} =$$
$$= (b_{a,n})^{\frac{a\cdot(j+c_0)}{a-m}} \cdot \left(\frac{r^m}{l_\mu^{s\cdot a}}\right)^{\frac{j}{a-m}} \cdot \left(\frac{1}{l_\mu^{s\cdot c_0}}\right)^{\frac{a}{a-m}} \cdot ||q||_{E\cap B(z_0,r)},$$

because $m < a$ and $j \le a \cdot n$. Specifically, for $j = a$ we have

$$a! \cdot |p'(z_0)|^a = |q^{(a)}(z_0)| \le (b_{a,n})^{\frac{a\cdot(a+c_0)}{a-m}} \cdot \left(\frac{r^m}{l_\mu^{s\cdot a}}\right)^{\frac{a}{a-m}} \cdot \left(\frac{1}{l_\mu^{s\cdot c_0}}\right)^{\frac{a}{a-m}} \cdot ||q||_{E\cap B(z_0,r)} \le$$
$$\le (b_{a,n})^{\frac{a\cdot(a+c_0)}{a-m}} \cdot \left(\frac{r^m}{l_\mu^{s\cdot(a+c_0)}}\right)^{\frac{a}{a-m}} \cdot 2^a \cdot \left(||p||_{E\cap B(z_0,r)}\right)^a,$$

because

$$q(z) = \left(p(z) - p(z_0)\right)^a = \left(\sum_{\nu=1}^{n} \frac{1}{\nu!} \cdot p^{(\nu)}(z_0) \cdot (z - z_0)^\nu\right)^a = \left(p'(z_0) \cdot (z - z_0)\right)^a + O\left(|z - z_0|^{a+1}\right).$$

From this it finally follows that

$$|p'(z_0)| \le (b_{a,n})^{\frac{a+c_0}{a-m}} \cdot \left(\frac{r^m}{l_\mu^{s\cdot(a+c_0)}}\right)^{\frac{1}{a-m}} \cdot 2 \cdot ||p||_{E\cap B(z_0,r)},$$

which implies inequality (6). $\square$

We have now completed the proof of the first part of our main result, which is the equivalence of LMP and SPH, albeit with some deterioration of the constants.

THEOREM 7.10 [cf. Bos-Milman, theorem A]. *For any compact set $E \subset\subset \mathbb{C}$ and $m, k \ge 1$ we have the following string of implications:*

$$\mathrm{LMP}(m,k) \implies \mathrm{SPW}(m,k) \implies \mathrm{SPQ}(m,k) \implies$$
$$\implies \mathrm{SPQH}(m,0,k) \implies \mathrm{SPH}(m,m,k+5m) \implies \mathrm{LMP}(m',k')$$

for any $m' > m$ and $k' > k + 8m$.

Corollary 7.11 [cf. Bos-Milman, theorem A]. *If in theorem 7.9 we assume that* $s = 1$ *and* $c_0 = 0$ *then, regardless of the choice of the integer* $a > m$*, we have* $m_a = 1$ *and therefore in the assertion we can take* $m' = 1$*. Consequently we have*

$$\begin{gathered}\mathrm{LMP}(1,k) \implies \mathrm{SPW}(1,k) \implies \mathrm{SPQ}(1,k) \implies \\ \implies \mathrm{SPQH}(1,0,k) \implies \mathrm{SPH}(1,1,k+5) \text{ with } c_0 = 0 \implies \mathrm{LMP}(1,k')\end{gathered}$$

for any $k' > k + 8$.

Analogous statements are true for the weak versions of these properties.

CHAPTER VIII

JACKSON PROPERTY (JP)

Definition 8.1 [Siciak 3; cf. Pleśniak 1; cf. Zerner]. For a compact set $E \subset\subset \mathbb{C}$ we define the space of functions on E, which can be rapidly approximated by holomorphic polynomials:

$$s(E) := \{f \in \mathcal{C}(E) \quad : \quad \forall \ell \geq 0 \quad \lim_{n\to\infty} n^{\ell} \cdot \operatorname{dist}_E(f, \mathcal{P}_n) = 0\}$$

with the following Jackson norms:

$$|f|_{\ell} := \|f\|_E + \sup_{n\in\mathbb{N}} n^{\ell} \cdot \operatorname{dist}_E(f, \mathcal{P}_n) \quad \text{for } \ell \geq 0,$$

$$|f|_{-1} := \|f\|_E.$$

Definition 8.2. A compact set $E \subset\subset \mathbb{C}$ admits the Jackson Property JP(s, v), where $s, v \geq 1$, if $\mathcal{H}^{\infty}(E)_{|E} \subset s(E)$ and

$$\exists c_0 \geq 0 \quad \forall \ell \geq 1 \quad \exists c_{\ell} \geq 1 \quad \forall 0 < \delta \leq 1 \quad \forall f \in \mathcal{H}^{\infty}(E_{\delta}) \quad : \quad |f_{|E}|_{\ell} \leq \left(\frac{c_{\ell}}{\delta^s}\right)^{\ell + c_0} \cdot ||f||_{E_{\delta}}$$

and additionally $c_{\ell} \leq c_1 \cdot \ell^v$. Without the last assumption we speak of the Weak Jackson Property WJP(s). We will write that the set E admits JP, respectively WJP, if it admits JP(s, v), respectively WJP(s), for some $s, v \geq 1$.

Definition 8.3. For a fixed compact set $E \subset\subset \mathbb{C}$ and $\zeta \notin E$ put $f_{\zeta}(z) := \frac{1}{\zeta - z}$ in some open neighbourhood of the set E and extend it to a function of class $\mathcal{C}^{\infty}(\mathbb{C})$ so that $f_{\zeta} \in \mathcal{H}^{\infty}(E)$.

Remark 8.4. Note that if $\mathcal{H}^{\infty}(E)_{|E} \subset s(E)$ then the set E must be polynomially convex. Indeed if we assume the contrary, then there exists a point $\zeta \in \hat{E} \setminus E$. Now construct a sequence of polynomials of best approximation for the function $f_{\zeta} \in \mathcal{H}^{\infty}(E)$ as in definition 8.3, i.e. $p_n \in \mathcal{P}_n$, such that $\|f_{\zeta} - p_n\|_E = \operatorname{dist}_E(f_{\zeta}, \mathcal{P}_n)$, and subsequently let $q_n(z) := 1 - (\zeta - z) \cdot p_n(z)$ so that $q_n \in \mathcal{P}_{n+1}$. We then see that for all $z \in E$ and $n \in \mathbb{N}$ we have

$$|q_n(z)| = |\zeta - z| \cdot |f_{\zeta}(z) - p_n(z)| \leq \operatorname{diam} E \cdot \|f_{\zeta} - p_n\|_E = \operatorname{diam} E \cdot \operatorname{dist}_E(f_{\zeta}, \mathcal{P}_n)$$

and consequently, by the definition of the polynomial hull,

$$1 = |q_n(\zeta)| \leq \|q_n\|_{\hat{E}} = \|q_n\|_E \leq \operatorname{diam} E \cdot \operatorname{dist}_E(f_{\zeta}, \mathcal{P}_n).$$

This demonstrates that on the set E it is not possible to approximate the function f_{ζ} using holomorphic polynomials, not to speak of rapid approximation.

We are now going to look for ways to identify sets admitting the Jackson Property.

Theorem 8.5 Jackson's theorem [cf. Bos-Milman, lemma 4.17; cf. Pleśniak 4; cf. Cheney; cf. Timan; cf. Jackson]. *For each interval $I = [a, b] \subset \mathbb{R}$, $a < b$, there exists a constant $C := \max\left\{1, \frac{\pi \cdot e \cdot (b-a)}{4}\right\}$ such that*

$$\forall f \in \mathcal{C}^{\infty}(\mathbb{C}) \quad \forall \ell \in \mathbb{N} \quad \forall n \in \mathbb{N} \quad : \quad \operatorname{dist}_I(f, \mathcal{P}_n) \leq \left(\frac{C \cdot \ell}{n}\right)^{\ell} \cdot \|f\|_{I,\ell}.$$

PROOF. Fix a complex smooth function $f \in \mathcal{C}^\infty(\mathbb{C})$ and $\ell, n \in \mathbb{N}$. We assume that $n \geq \ell$ since otherwise the assertion is trivial, because in such case

$$\operatorname{dist}_I(f, \mathcal{P}_n) \leq \|f\|_I \leq \left(\frac{C \cdot \ell}{n}\right)^\ell \cdot \|f\|_{I,\ell}.$$

We perform a linear change of the variable and we split the function f into its real and imaginary parts, i.e.

$$f_1(x) := \Re f\left(\frac{a+b+x\cdot(b-a)}{2}\right),$$
$$f_2(x) := \Im f\left(\frac{a+b+x\cdot(b-a)}{2}\right),$$

so that $f_1(x) + f_2(x)\cdot i = f\left(\frac{a+b+x\cdot(b-a)}{2}\right)$ for $x \in \mathbb{R}$ and $f_1, f_2 \in \mathcal{C}^\infty(\mathbb{R})$ are real smooth functions.

By the classic Jackson theorems [Jackson, Timan, Cheney or Pleśniak 4] we have

$$\operatorname{dist}_{[-1,1]}\Big(f_1, \mathcal{P}_n(\mathbb{R})\Big) \leq \frac{\left(\frac{\pi}{2}\right)^\ell \cdot \left\|f_1^{(\ell)}\right\|_{[-1,1]}}{(n+1)\cdot n \cdot \ldots \cdot (n-\ell+2)}.$$

Therefore we can find a polynomial $p_1 \in \mathcal{P}_n(\mathbb{R})$ with real coefficients for which we have

$$n^\ell \cdot \|f_1 - p_1\|_{[-1,1]} \leq \left(\frac{\pi}{2}\right)^\ell \cdot \frac{n^\ell}{(n+1)\cdot n\cdot\ldots\cdot(n-\ell+2)} \cdot \left\|f_1^{(\ell)}\right\|_{[-1,1]} =$$
$$= \left(\frac{\pi}{2}\right)^\ell \cdot \frac{n}{n+1} \cdot \frac{n^{\ell-1}}{n\cdot\ldots\cdot(n-\ell+2)} \cdot \left\|f_1^{(\ell)}\right\|_{[-1,1]} \leq \left(\frac{\pi}{2}\right)^\ell \cdot \frac{\ell^{\ell-1}}{\ell!} \cdot \left\|f_1^{(\ell)}\right\|_{[-1,1]},$$

because the expression $\frac{n^{\ell-1}}{n\cdot\ldots\cdot(n-\ell+2)}$ (interpreted as 1 when $\ell = 1$) diminishes when $n \geq \ell$ increases. By Stirling's formula we have $\ell! \geq \left(\frac{\ell}{e}\right)^\ell \cdot \sqrt{2\pi\ell}$ and consequently

$$\frac{\ell^{\ell-1}}{\ell!} \leq \frac{e^\ell}{\ell\cdot\sqrt{2\pi\ell}},$$

which implies that

$$n^\ell \cdot \|f_1 - p_1\|_{[-1,1]} \leq \left(\frac{\pi\cdot e}{2}\right)^\ell \cdot \frac{1}{\ell\cdot\sqrt{2\pi\ell}} \cdot \left\|f_1^{(\ell)}\right\|_{[-1,1]} \leq \left(\frac{\pi\cdot e}{2}\right)^\ell \cdot \frac{1}{\sqrt{2\pi}} \cdot \left\|f_1^{(\ell)}\right\|_{[-1,1]}.$$

Note that because f_1 and f_2 and their derivatives are real functions, we have for $x \in [-1,1]$

$$\left|f_1^{(\ell)}(x)\right| \leq \left|f_1^{(\ell)}(x) + f_2^{(\ell)}(x)\cdot i\right| = \left|\left(\frac{b-a}{2}\right)^\ell \cdot \frac{\partial^\ell f}{\partial z^\ell}\left(\frac{a+b+x\cdot(b-a)}{2}\right)\right| \leq \left(\frac{b-a}{2}\right)^\ell \cdot \left\|\frac{\partial^\ell f}{\partial z^\ell}\right\|_I$$

and therefore

$$\|f_1 - p_1\|_{[-1,1]} \leq \left(\frac{\pi\cdot e\cdot(b-a)}{4n}\right)^\ell \cdot \frac{1}{\sqrt{2\pi}} \cdot \left\|\frac{\partial^\ell f}{\partial z^\ell}\right\|_I.$$

Similarly there exists a polynomial $p_2 \in \mathcal{P}_n(\mathbb{R})$ such that

$$\|f_2 - p_2\|_{[-1,1]} \leq \left(\frac{\pi\cdot e\cdot(b-a)}{4n}\right)^\ell \cdot \frac{1}{\sqrt{2\pi}} \cdot \left\|\frac{\partial^\ell f}{\partial z^\ell}\right\|_I$$

and finally we see that

$$\operatorname{dist}_I(f, \mathcal{P}_n) = \operatorname{dist}_{[-1,1]}(f_1 + f_2\cdot i, \mathcal{P}_n) \leq \|(f_1 + f_2\cdot i) - (p_1 + p_2\cdot i)\|_{[-1,1]} =$$
$$= \|(f_1 - p_1) + (f_2 - p_2)\cdot i)\|_{[-1,1]} \leq \|f_1 - p_1\|_{[-1,1]} + \|f_2 - p_2\|_{[-1,1]} \leq$$
$$\leq \left(\frac{\pi\cdot e\cdot(b-a)}{4n}\right)^\ell \cdot \left\|\frac{\partial^\ell f}{\partial z^\ell}\right\|_I \leq \left(\frac{C\cdot\ell}{n}\right)^\ell \cdot \|f\|_{I,\ell}. \quad \square$$

Corollary 8.6. *For any compact set $E \subset\subset \mathbb{R} \subset \mathbb{C}$ we have $\mathcal{C}^\infty(E) \subset s(E)$ and furthermore there exists a constant $\widetilde{C} \geq 1$ such, that*

$$\forall f \in \mathcal{C}^\infty(E) \quad \forall \ell \in \mathbb{N} \quad : \quad |f|_\ell \leq \left(\widetilde{C} \cdot \ell\right)^\ell \cdot |\!|\!| f |\!|\!|_{E,\ell}.$$

Proof. For the interval $I := \operatorname{conv} E$ we determine the constant C from Jackson's theorem 8.5. For $f \in \mathcal{C}^\infty(E)$ we consider an arbitrary extension $\widetilde{f} \in \mathcal{C}^\infty(\mathbb{C})$, such that $\widetilde{f}_{|E} = f$. Then for any $\ell \in \mathbb{N}$ we have

$$\sup_{n\in\mathbb{N}} n^\ell \cdot \operatorname{dist}_E(f, \mathcal{P}_n) = \sup_{n\in\mathbb{N}} n^\ell \cdot \operatorname{dist}_E(\widetilde{f}, \mathcal{P}_n) \leq \sup_{n\in\mathbb{N}} n^\ell \cdot \operatorname{dist}_I(\widetilde{f}, \mathcal{P}_n) \leq \left(C \cdot \ell\right)^\ell \cdot \|\widetilde{f}\|_{I,\ell} < +\infty.$$

This proves that $f \in s(E)$ but also, by taking the infimum over all possible extensions $\widetilde{f} \in \mathcal{C}^\infty(\mathbb{C})$, we obtain

$$|f|_\ell = \|f\|_E + \sup_{n\in\mathbb{N}} n^\ell \cdot \operatorname{dist}_E(f, \mathcal{P}_n) \leq$$
$$\leq \left(1 + \left(C \cdot \ell\right)^\ell\right) \cdot \inf\left\{\|\widetilde{f}\|_{I,\ell} \ : \ \widetilde{f} \in \mathcal{C}^\infty(\mathbb{C}),\ \widetilde{f}_{|E} = f\right\} \leq \left(\widetilde{C} \cdot \ell\right)^\ell \cdot |\!|\!| f |\!|\!|_{E,\ell},$$

where $\widetilde{C} := 1 + C$. $\square$

Corollary 8.7. *Every interval $I = [a, b] \subset \mathbb{R}$ admits* JP$(1, 2)$.

Proof. Fix $0 < \delta \leq 1$, $f \in \mathcal{H}^\infty(I_\delta)$ and $\ell \in \mathbb{N}$. Cauchy's integral formula implies that for each $z \in I$ we have

$$f^{(\ell)}(z) = \frac{\ell!}{2\pi i} \cdot \int_{\partial B(z,\delta)} \frac{f(\zeta)}{(\zeta - z)^{\ell+1}}\, d\zeta$$

and therefore

$$|f^{(\ell)}(z)| \leq \frac{\ell!}{2\pi} \cdot \frac{\|f\|_{I_\delta}}{\delta^{\ell+1}} \cdot 2\pi \cdot \delta = \frac{\ell!}{\delta^\ell} \cdot \|f\|_{I_\delta},$$
$$\|f\|_{I,\ell} = \|f\|_I + \|f^{(\ell)}\|_I \leq \left(1 + \frac{\ell!}{\delta^\ell}\right) \cdot \|f\|_{I_\delta} \leq \frac{2\ell^\ell}{\delta^\ell} \cdot \|f\|_{I_\delta}.$$

Consequently corollary 8.6 leads us to conclude that

$$|f_{|I}|_\ell \leq \left(\widetilde{C} \cdot \ell\right)^\ell \cdot |\!|\!| f |\!|\!|_{I,\ell} = \left(\widetilde{C} \cdot \ell\right)^\ell \cdot \ \|f\|_{I,\ell} \leq \left(\widetilde{C} \cdot \ell\right)^\ell \cdot \frac{2\ell^\ell}{\delta^\ell} \cdot \|f\|_{I_\delta} \leq \left(\frac{2\widetilde{C} \cdot \ell^2}{\delta}\right)^\ell \cdot \|f\|_{I_\delta}.$$

Finally for any $\ell \geq 1$, not necessarily integer, we obtain

$$|f_{|I}|_\ell \leq |f_{|I}|_{\operatorname{int}(\ell+1)} \leq \left(\frac{2\widetilde{C} \cdot (\ell+1)^2}{\delta}\right)^{\ell+1} \cdot \|f\|_{I_\delta} \leq \left(\frac{8\widetilde{C} \cdot \ell^2}{\delta}\right)^{\ell+1} \cdot \|f\|_{I_\delta}. \quad \square$$

Corollary 8.8. *Every compact set $E \subset\subset \mathbb{R} \subset \mathbb{C}$ admits* JP$(1, 6)$.

Proof. Fix $0 < \delta \leq 1$ and $f \in \mathcal{H}^\infty(E_\delta)$. Corollary 8.6 and proposition 6.12 imply that $\mathcal{H}^\infty(E) \subset \mathcal{C}^\infty(E) \subset s(E)$ and for all $\ell \in \mathbb{N}$ we have

$$|f_{|E}|_\ell \leq \left(\widetilde{C} \cdot \ell\right)^\ell \cdot |\!|\!| f |\!|\!|_{E,\ell} \leq \left(\widetilde{C} \cdot \ell\right)^\ell \cdot \langle\!\langle f \rangle\!\rangle_{E,\ell} \leq \left(\widetilde{C} \cdot \ell\right)^\ell \cdot \langle\!\langle f \rangle\!\rangle_{E_{\delta/17},\ell} \leq$$
$$\leq \frac{\left(\widetilde{C} \cdot \ell\right)^\ell \cdot \left(2d \cdot \ell\right)^{5\cdot\ell}}{\delta^\ell} \cdot ||f||_{E_\delta} \leq \left(\frac{\widetilde{C} \cdot 2^5 \cdot d^5 \cdot \ell^6}{\delta}\right)^\ell \cdot ||f||_{E_\delta},$$

where $d \geq 1$ is the absolute constant from proposition 6.8 on cutoff functions. Finally for any $\ell \geq 1$, not necessarily integer, we obtain

$$|f_{|E}|_\ell \leq |f_{|E}|_{\operatorname{int}(\ell+1)} \leq \left(\frac{\widetilde{C} \cdot 2^5 \cdot d^5 \cdot (\ell+1)^6}{\delta}\right)^{\ell+1} \cdot ||f||_{E_\delta} \leq \left(\frac{\widetilde{C} \cdot 2^{11} \cdot d^5 \cdot \ell^6}{\delta}\right)^{\ell+1} \cdot ||f||_{E_\delta}. \quad \square$$

LEMMA 8.9. *For $n, \ell \in \mathbb{N}$, $n \geq \ell$, put*

$$\varphi_{n,\ell} := n^\ell \cdot \sum_{j=n-\ell}^{\infty} \frac{j!}{(j+\ell+1)!}.$$

Then for each such n, ℓ we have $\varphi_{n,\ell} \leq \frac{\ell^\ell}{\ell \cdot \ell!}$.

PROOF. Note that

$$\frac{j!}{(j+\ell)!} - \frac{(j+1)!}{(j+\ell+1)!} = \frac{j! \cdot (j+\ell+1) - j! \cdot (j+1)}{(j+\ell+1)!} = \frac{j! \cdot \ell}{(j+\ell+1)!}$$

and consequently

$$\varphi_{n,\ell} = n^\ell \cdot \sum_{j=n-\ell}^{\infty} \frac{1}{\ell} \cdot \left(\frac{j!}{(j+\ell)!} - \frac{(j+1)!}{(j+\ell+1)!} \right) = \frac{n^\ell}{\ell} \cdot \frac{(n-\ell)!}{n!}.$$

The latter expression decreases when n increases and therefore it attains its maximum when $n = \ell$. $\square$

PROPOSITION 8.10. JACKSON'S THEOREM FOR THE COMPLEX BALL. *Consider a ball $B := B(z_0, r)$, where $z_0 \in \mathbb{C}$, $r > 0$, and an arbitrary function $f \in \mathcal{A}^\infty(B)$. Then we have $f_{|B} \in s(B)$ and furthermore the Jackson norms of the restriction $f_{|B}$ can be estimated as follows:*

$$\forall \ell \in \mathbb{N} \quad : \quad |f_{|B}|_\ell \leq (c \cdot \ell)^{\ell+1} \cdot \|f\|_{B,\ell+1},$$

where $c := \max\{2, r\}$. Additionally, provided that $f \in \mathcal{H}^\infty(B_\delta)$ with some $0 < \delta \leq 1$, we have

$$\forall \ell \geq 1 \quad : \quad |f_{|B}|_\ell \leq \left(\frac{c \cdot \ell}{\delta} \right)^{\ell+1} \cdot \|f\|_{B_\delta}.$$

PROOF. Fix $\ell \in \mathbb{N}$. By developing the function f into a Taylor series around the point z_0 we obtain for all each point $z \in \operatorname{int} B$ in the interior of the ball B

$$f(z) = \sum_{j=0}^{\infty} a_j \cdot (z - z_0)^j,$$

$$f^{(\ell+1)}(z) = \sum_{j=\ell+1}^{\infty} a_j \cdot \frac{j!}{(j-\ell-1)!} \cdot (z-z_0)^{j-\ell-1} = \sum_{j=0}^{\infty} a_{j+\ell+1} \cdot \frac{(j+\ell+1)!}{j!} \cdot (z-z_0)^j.$$

We put $\epsilon := 0$ and apply Cauchy's integral formula to both series to obtain

$$a_j = \frac{1}{2\pi i} \cdot \int_{\partial B_\epsilon} \frac{f(\zeta)}{(\zeta - z_0)^{j+1}} \, d\zeta,$$

$$a_{j+\ell+1} \cdot \frac{(j+\ell+1)!}{j!} = \frac{1}{2\pi i} \cdot \int_{\partial B} \frac{f^{(\ell+1)}(\zeta)}{(\zeta - z_0)^{j+1}} \, d\zeta.$$

This implies that for each $j \in \mathbb{N}$ we have

$$(1) \qquad |a_j| \leq \frac{1}{2\pi} \cdot \int_{\partial B_\epsilon} \frac{\|f\|_{B_\epsilon}}{(r+\epsilon)^{j+1}} \, |d\zeta| = \frac{\|f\|_{B_\epsilon}}{(r+\epsilon)^j},$$

$$(2) \qquad |a_{j+\ell+1}| \leq \frac{j!}{(j+\ell+1)!} \cdot \frac{1}{2\pi} \cdot \int_{\partial B} \frac{\|f^{(\ell+1)}\|_B}{r^{j+1}} \, |d\zeta| = \frac{j!}{(j+\ell+1)!} \cdot \frac{\|f^{(\ell+1)}\|_B}{r^j}.$$

Next put $p_n(z) := \sum_{j=0}^{n} a_j \cdot (z-z_0)^j$, $p_n \in \mathcal{P}_n$, and consider the remainder $s_n(z) := f(z) - p_n(z)$, $s_n \in \mathcal{A}^\infty(B)$. Obviously for each $z \in \operatorname{int} B$ we have $|s_n(z)| \le \sum_{j=n+1}^{\infty} |a_j| \cdot r^j$ so this estimate must hold on the boundary ∂B too. Therefore we can estimate the Jackson norms as follows:

$$|f_{|B}|_\ell = \|f\|_B + \sup_{n\in\mathbb{N}} n^\ell \cdot \operatorname{dist}_B(f, \mathcal{P}_n) \le \|f\|_B + \sup_{n\in\mathbb{N}} n^\ell \cdot \|s_n\|_B \le \|f\|_B + \sup_{n\in\mathbb{N}} \left(n^\ell \cdot \sum_{j=n+1}^{\infty} |a_j| \cdot r^j \right).$$

If $n \ge \ell$ then we can apply inequality (2) and lemma 8.9 to obtain

$$n^\ell \cdot \sum_{j=n+1}^{\infty} |a_j| \cdot r^j = n^\ell \cdot \sum_{j=n-\ell}^{\infty} |a_{j+\ell+1}| \cdot r^{j+\ell+1} \le n^\ell \cdot \sum_{j=n-\ell}^{\infty} \frac{j!}{(j+\ell+1)!} \cdot \|f^{(\ell+1)}\|_B \cdot r^{\ell+1} =$$
$$= \varphi_{n,\ell} \cdot \|f^{(\ell+1)}\|_B \cdot r^{\ell+1} \le r \cdot (r\cdot\ell)^\ell \cdot \|f^{(\ell+1)}\|_B.$$

If $n < \ell$ then we use inequalities (1) with $\epsilon = 0$ and (2) to obtain

$$n^\ell \cdot \sum_{j=n+1}^{\infty} |a_j| \cdot r^j = n^\ell \cdot \left(\sum_{j=n+1}^{\ell} |a_j| \cdot r^j + \sum_{j=0}^{\infty} |a_{j+\ell+1}| \cdot r^{j+\ell+1} \right) \le$$
$$\le \ell^\ell \cdot \left(\sum_{j=n+1}^{\ell} \|f\|_B + \sum_{j=0}^{\infty} \frac{j!}{(j+\ell+1)!} \cdot \|f^{(\ell+1)}\|_B \cdot r^{\ell+1} \right) \le$$
$$\le \ell^\ell \cdot \ell \cdot \|f\|_B + \varphi_{\ell,\ell} \cdot \|f^{(\ell+1)}\|_B \cdot r^{\ell+1} \le \ell^{\ell+1} \cdot \|f\|_B + r \cdot (r\cdot\ell)^\ell \cdot \|f^{(\ell+1)}\|_B.$$

In either case we conclude that

$$|f_{|B}|_\ell \le \left(1 + \ell^{\ell+1}\right) \cdot \|f\|_B + r \cdot (r\cdot\ell)^\ell \cdot \|f^{(\ell+1)}\|_B \le$$
$$\le (c\cdot\ell)^{\ell+1} \cdot \left(\|f\|_B + \|f^{(\ell+1)}\|_B \right) = (c\cdot\ell)^{\ell+1} \cdot \|f\|_{B,\ell+1} < +\infty.$$

Additionally, if $f \in \mathcal{H}^\infty(B_\delta)$ with some $0 < \delta \le 1$, then we use Cauchy's integral formula to obtain inequality (1) with $\epsilon := \delta$. Consequently we have

$$\sum_{j=n+1}^{\infty} |a_j| \cdot r^j \le \sum_{j=n+1}^{\infty} \left(\frac{r}{r+\delta} \right)^j \cdot \|f\|_{B_\delta} = \left(\frac{r}{r+\delta} \right)^{n+1} \cdot \frac{1}{1 - \frac{r}{r+\delta}} \cdot \|f\|_{B_\delta} = \left(\frac{r}{r+\delta} \right)^{n} \cdot \frac{r}{\delta} \cdot \|f\|_{B_\delta},$$

$$\sup_{n\in\mathbb{N}} \left(n^\ell \cdot \sum_{j=n+1}^{\infty} |a_j| \cdot r^j \right) \le \sup_{n\in\mathbb{R}_+} \left(n^\ell \cdot \left(\frac{r}{r+\delta} \right)^{n} \cdot \frac{r}{\delta} \cdot \|f\|_{B_\delta} \right) = \left(\frac{\ell}{e \cdot \log \frac{r+\delta}{r}} \right)^\ell \cdot \frac{r}{\delta} \cdot \|f\|_{B_\delta}$$

because for $a, b > 0$

$$\sup_{n\in\mathbb{R}_+} n^a \cdot e^{-b\cdot n} = \left(\frac{a}{b\cdot e} \right)^a.$$

Since for each $x \ge 1$ we have $\left(1 + \frac{1}{r\cdot x}\right)^x \ge 1 + \frac{1}{r}$, by taking $x = \frac{1}{\delta}$ we see that $1 + \frac{\delta}{r} \ge \left(1 + \frac{1}{r}\right)^\delta$ and it is also easy to verify that $\log \frac{r+\delta}{r} \ge \delta \cdot \log(1 + \frac{1}{r}) \ge \frac{4}{5} \cdot \delta \cdot \min\{\frac{1}{2}, \frac{1}{r}\}$. This implies that

$$|f_{|B}|_\ell \le \|f\|_B + \left(\frac{\ell}{e \cdot \log \frac{r+\delta}{r}} \right)^\ell \cdot \frac{r}{\delta} \cdot \|f\|_{B_\delta} \le \|f\|_B + \left(\frac{\ell}{e \cdot \frac{4}{5} \cdot \delta \cdot \min\left\{\frac{1}{2}, \frac{1}{r}\right\}} \right)^\ell \cdot \frac{r}{\delta} \cdot \|f\|_{B_\delta} =$$
$$= \|f\|_B + \left(\frac{5c\cdot\ell}{4e\cdot\delta} \right)^\ell \cdot \frac{r}{\delta} \cdot \|f\|_{B_\delta} \le \|f\|_B + \left(\frac{\frac{1}{2}c\cdot\ell}{\delta} \right)^\ell \cdot \frac{c}{\delta} \cdot \|f\|_{B_\delta} \le \left(1 + \left(\frac{\frac{1}{2}c\cdot\ell}{\delta} \right)^\ell \right) \cdot \frac{c}{\delta} \cdot \|f\|_{B_\delta} \le$$
$$\le \left(\frac{(1+\frac{1}{2}c)\cdot\ell}{\delta} \right)^\ell \cdot \frac{c}{\delta} \cdot \|f\|_{B_\delta} \le \left(\frac{c\cdot\ell}{\delta} \right)^\ell \cdot \frac{c}{\delta} \cdot \|f\|_{B_\delta} \le \left(\frac{c\cdot\ell}{\delta} \right)^{\ell+1} \cdot \|f\|_{B_\delta}. \quad \square$$

COROLLARY 8.11. *Every ball admits* JP$(1,1)$.

Our goal will now be to establish a general version of Jackson's theorem in the complex plane.

LEMMA 8.12. *For all* $\zeta \notin E \subset\subset \mathbb{C}$ *and* $n \in \mathbb{Z}_+$ *we have*

$$\frac{1}{\Phi_{n+1}(\zeta)\cdot\big(\operatorname{dist}(\zeta,E)+\operatorname{diam}E\big)} \le \operatorname{dist}_E(f_\zeta,\mathcal{P}_n) \le \frac{1}{\Phi_{n+1}(\zeta)\cdot\operatorname{dist}(\zeta,E)}.$$

PROOF. Fix $n \in \mathbb{N}$ and take an arbitrary polynomial $q \in \mathcal{P}_{n+1}$ such that $\|q\|_E = 1$ and $q(\zeta) \neq 0$. Define $p(z) := \frac{q(\zeta)-q(z)}{q(\zeta)\cdot(\zeta-z)}$ so that $p \in \mathcal{P}_n$. Then we obtain

$$\operatorname{dist}_E(f_\zeta,\mathcal{P}_n) \le \|f_\zeta - p\|_E = \sup_{z\in E}|f_\zeta(z)-p(z)| =$$
$$= \sup_{z\in E}\left|\frac{q(z)}{q(\zeta)\cdot(\zeta-z)}\right| \le \frac{\|q\|_E}{|q(\zeta)|\cdot\inf_{z\in E}|\zeta-z|} = \frac{1}{|q(\zeta)|\cdot\operatorname{dist}(\zeta,E)}.$$

We take the infimum over such $q \in \mathcal{P}_{n+1}$ to arrive at $\operatorname{dist}_E(f_\zeta,\mathcal{P}_n) \le \frac{1}{\Phi_{n+1}(\zeta)\cdot\operatorname{dist}(\zeta,E)}$.

On the other hand for fixed $n \in \mathbb{N}$ find $p \in \mathcal{P}_n$ such that $\operatorname{dist}_E(f_\zeta,\mathcal{P}_n) = \|f_\zeta - p\|_E$. Define $q(z) := 1 - p(z)\cdot(\zeta - z)$ so that $q \in \mathcal{P}_{n+1}$. We see that

$$\|q\|_E = \sup_{z\in E}|1-p(z)\cdot(\zeta-z)| \le \sup_{z\in E}|\zeta-z|\cdot\sup_{z\in E}|f_\zeta(z)-p(z)| \le \big(\operatorname{dist}(\zeta,E)+\operatorname{diam}E\big)\cdot\operatorname{dist}_E(f_\zeta,\mathcal{P}_n)$$

and hence

$$\Phi_{n+1}(\zeta) \ge \frac{|q(\zeta)|}{\|q\|_E} \ge \frac{1}{\big(\operatorname{dist}(\zeta,E)+\operatorname{diam}E\big)\cdot\operatorname{dist}_E(f_\zeta,\mathcal{P}_n)}. \qquad \square$$

DEFINITION 8.13. For a fixed compact set $E \subset\subset \mathbb{C}$, $n \in \mathbb{Z}_+$ and $0 \le t < +\infty$ let

$$\phi_n(t) := \inf_{z\in\partial E_t}\Phi_n(z),$$
$$\phi_E(t) := \inf_{z\in\partial E_t}\Phi_E(z).$$

Here and further in this chapter we denote by ∂E_t the set $\{z \in \mathbb{C} \;:\; \operatorname{dist}(z,E) = t\}$, i.e. the boundary of the interior of the set E_t, which can be a slightly bigger set than only the boundary of E_t.

REMARK 8.14. Note that for $t > 0$ the functions ϕ_n and ϕ_E are continuous or equal to $+\infty$. Furthermore we have for all $n_0 \le n$ and $t_0 \le t$

$$\phi_{n_0}(t_0) \le \phi_{n_0}(t) \le \phi_n(t) \le \big(\phi_E(t)\big)^n,$$

where the first inequality follows from the maximum principle for subharmonic functions, applied to the function $\log\Phi_{n_0}$, and the latter two inequalities follow straight from the definitions.

DEFINITION 8.15. For a compact set $E \subset\subset \mathbb{C}$ and $\delta > 0$ denote by $K(E,\delta)$ a compact neighbourhood constructed as follows. First we cut up the entire complex plane into closed squares of size $\delta\times\delta$, starting at the origin of the plane. Next we select all squares having a non-empty intersection with the set E and by $K(E,\delta)$ we denote the sum of those squares.

Clearly we have $E \subset K(E,\delta) \subset E_{\delta\cdot\sqrt{2}}$. Also it is easy to see that the set $K(E,\delta)$ consists of at most $\big(\frac{\operatorname{diam}E}{\delta}+2\big)^2$ squares and therefore the length of its border $\partial K(E,\delta)$ is at most

$$\left(\frac{\operatorname{diam}E}{\delta}+2\right)^2\cdot 4\delta = \frac{4\cdot(\operatorname{diam}E_\delta)^2}{\delta}.$$

The following results were inspired by the proof of Runge's theorem as given by [Gaier, chapter II §3 and chapter III §1]. They show that there is a direct relationship between the approximation of holomorphic functions on a compact set and the behavior of its extremal function.

PROPOSITION 8.16. *For any compact set $E \subset\subset \mathbb{C}$, $0<\delta\le 1$ and $f\in\mathcal{H}^\infty(E_\delta)$ we have*

$$\forall \frac{1}{2}\le b<1 \quad \forall n\in\mathbb{N} \quad : \quad \operatorname{dist}_E(f,\mathcal{P}_n)\le \frac{c\cdot\|f\|_{E_\delta}}{(1-b)\cdot\delta^2\cdot\phi_{n+1}(b\cdot\delta)},$$

where the constant $c:=\frac{28}{\pi}\cdot(2+\operatorname{diam}E)^2$ depends only on the set E.

PROOF. Fix $\frac{1}{2}\le b<1$ and $n\in\mathbb{N}$. If $\phi_{n+1}(b\cdot\delta)=+\infty$ then the set E consists of $n+1$ or less points and $\operatorname{dist}_E(f,\mathcal{P}_n)=0$, which finishes the proof. Otherwise find a $\widetilde{\delta}$ such that the number $\frac{(1-b)\cdot\delta}{4\cdot\widetilde{\delta}}$ is an integer and furthermore

$$0<\widetilde{\delta}\le\frac{(1-b)\cdot\delta}{4\cdot\phi_{n+1}(b\cdot\delta)}.$$

Let Γ be the boundary $\partial K\left(E_{b\cdot\delta},\frac{1-b}{4}\cdot\delta\right)$, with proper orientation, and cut it up into equal intervals Γ_j, each of length $\widetilde{\delta}$, so that $\Gamma=\bigcup_j\Gamma_j$, with j running over a finite index set. As $K\left(E_{b\cdot\delta},\frac{1-b}{4}\cdot\delta\right)\subset E_{b\cdot\delta+\frac{1-b}{4}\cdot\delta\cdot\sqrt{2}}\subset E_{\frac{1+b}{2}\cdot\delta}$ we see that $\Gamma\subset E_{\frac{1+b}{2}\cdot\delta}\setminus\operatorname{int}E_{b\cdot\delta}$ while for the length of Γ, denoted $m(\Gamma)$, we have

$$\sum_j\widetilde{\delta}=m(\Gamma)\le\frac{4\cdot(\operatorname{diam}E_\delta)^2}{\frac{1-b}{4}\cdot\delta}\le\frac{4\pi\cdot c}{7\cdot(1-b)\cdot\delta}.$$

For $z\in E$ put $g_z(\zeta):=\frac{f(\zeta)}{\zeta-z}$, which is a holomorphic function in some open neighbourhood of the set $E_\delta\setminus\{z\}$. Let $\zeta_0,\zeta_1\in\Gamma_j$ for some j. Then the entire interval $I:=[\zeta_0,\zeta_1]$ lies in Γ_j and of course $\operatorname{dist}(z,I)\ge b\cdot\delta$. Since $g_z'(\zeta)=\frac{f'(\zeta)}{\zeta-z}-\frac{f(\zeta)}{(\zeta-z)^2}$ we have for all $\zeta\in I$, thanks to Cauchy's integral formula applied to $f'\in\mathcal{H}^\infty(E_\delta)$

$$|g_z'(\zeta)|\le\frac{\|f'\|_{E_{\frac{1+b}{2}\cdot\delta}}}{b\cdot\delta}+\frac{\|f\|_{E_{\frac{1+b}{2}\cdot\delta}}}{(b\cdot\delta)^2}\le\frac{\|f\|_{E_\delta}}{\frac{1-b}{2}\cdot b\cdot\delta^2}+\frac{\|f\|_{E_\delta}}{(b\cdot\delta)^2}=\frac{(1+b)\cdot\|f\|_{E_\delta}}{(1-b)\cdot b^2\cdot\delta^2}\le\frac{6\cdot\|f\|_{E_\delta}}{(1-b)\cdot\delta^2}.$$

This leads us to

$$\left|\frac{f(\zeta_1)}{\zeta_1-z}-\frac{f(\zeta_0)}{\zeta_0-z}\right|=|g_z(\zeta_1)-g_z(\zeta_0)|=\left|\int_I g_z'(\zeta)\,d\zeta\right|\le\int_I|g_z'(\zeta)|\,|d\zeta|\le$$
$$\le\frac{6\cdot\|f\|_{E_\delta}}{(1-b)\cdot\delta^2}\cdot|\zeta_1-\zeta_0|\le\frac{6\cdot\|f\|_{E_\delta}}{(1-b)\cdot\delta^2}\cdot\widetilde{\delta}\le\frac{6\cdot\|f\|_{E_\delta}}{4\delta\cdot\phi_{n+1}(b\cdot\delta)}.$$

We now see that for all $z\in E$, all j and arbitrarily selected points $\zeta_j\in\Gamma_j$ we have

$$\left|\frac{1}{2\pi i}\cdot\int_{\Gamma_j}\frac{f(\zeta)}{\zeta-z}\,d\zeta-\frac{1}{2\pi i}\cdot\int_{\Gamma_j}\frac{f(\zeta_j)}{\zeta_j-z}\,d\zeta\right|\le\frac{1}{2\pi}\cdot\int_{\Gamma_j}\left|\frac{f(\zeta)}{\zeta-z}-\frac{f(\zeta_j)}{\zeta_j-z}\right|\,|d\zeta|\le$$
$$\le\frac{1}{2\pi}\cdot\int_{\Gamma_j}\frac{6\cdot\|f\|_{E_\delta}}{4\delta\cdot\phi_{n+1}(b\cdot\delta)}\,|d\zeta|=\frac{3\cdot\|f\|_{E_\delta}\cdot\widetilde{\delta}}{4\pi\cdot\delta\cdot\phi_{n+1}(b\cdot\delta)}.$$

By summing over j we obtain

$$\left|\frac{1}{2\pi i}\cdot\int_\Gamma\frac{f(\zeta)}{\zeta-z}\,d\zeta-R(z)\right|\le\sum_j\frac{3\cdot\widetilde{\delta}\cdot\|f\|_{E_\delta}}{4\pi\cdot\delta\cdot\phi_{n+1}(b\cdot\delta)}=\frac{3\cdot m(\Gamma)\cdot\|f\|_{E_\delta}}{4\pi\cdot\delta\cdot\phi_{n+1}(b\cdot\delta)},$$

where we denote

$$R(z):=\sum_j\left(\frac{1}{2\pi i}\cdot\int_{\Gamma_j}\frac{f(\zeta_j)}{\zeta_j-z}\,d\zeta\right)=\sum_j\frac{c_j}{\zeta_j-z}=\sum_j c_j\cdot f_{\zeta_j}(z),$$

where in turn $c_j:=\frac{1}{2\pi i}\cdot f(\zeta_j)\cdot\int_{\Gamma_j}d\zeta$ and hence $|c_j|\le\frac{1}{2\pi}\cdot\|f\|_{E_\delta}\cdot\widetilde{\delta}$. Because by Cauchy's integral formula we have $f(z)=\frac{1}{2\pi i}\cdot\int_\Gamma\frac{f(\zeta)}{\zeta-z}\,d\zeta$ for all $z\in E$, we now have found a rational function R, which approximates the function f uniformly on the set E (and actually in a certain large neighbourhood as well):

$$\|f-R\|_E \le \frac{3\cdot m(\Gamma)\cdot \|f\|_{E_\delta}}{4\pi\cdot\delta\cdot\phi_{n+1}(b\cdot\delta)}.$$

Simultaneously by virtue of lemma 8.12 and by the minimum principle we have

$$\operatorname{dist}_E(R,\mathcal{P}_n) \le \sum_j |c_j|\cdot \operatorname{dist}_E(f_{\zeta_j},\mathcal{P}_n) \le \sum_j \frac{\|f\|_{E_\delta}\cdot\widetilde{\delta}}{2\pi\cdot\Phi_{n+1}(\zeta_j)\cdot\operatorname{dist}(\zeta_j,E)} \le$$
$$\le \sum_j \frac{\|f\|_{E_\delta}\cdot\widetilde{\delta}}{2\pi\cdot\phi_{n+1}(b\cdot\delta)\cdot b\cdot\delta} = \frac{\|f\|_{E_\delta}\cdot m(\Gamma)}{2\pi\cdot\phi_{n+1}(b\cdot\delta)\cdot b\cdot\delta},$$

because $\operatorname{dist}(\zeta_j,E)\ge b\cdot\delta$. Consequently, since $\frac{1}{b}\le 2$, we conclude that

$$\operatorname{dist}_E(f,\mathcal{P}_n) \le \|f-R\|_E + \operatorname{dist}_E(R,\mathcal{P}_n) \le \frac{7\cdot m(\Gamma)\cdot\|f\|_{E_\delta}}{4\pi\cdot\delta\cdot\phi_{n+1}(b\cdot\delta)} \le \frac{c\cdot\|f\|_{E_\delta}}{(1-b)\cdot\delta^2\cdot\phi_{n+1}(b\cdot\delta)}. \quad \square$$

Corollary 8.17. *For any compact set $E\subset\subset\mathbb{C}$, $0<\delta\le 1$ and $f\in\mathcal{H}^\infty(E_\delta)$ we have*

$$\limsup_{n\to\infty}\sqrt[n]{\operatorname{dist}_E(f,\mathcal{P}_n)} \le \frac{1}{\phi_E(\delta)}.$$

Proof. Proposition 8.16 implies that for any $\frac{1}{2}\le b<1$ we have

$$\limsup_{n\to\infty}\sqrt[n]{\operatorname{dist}_E(f,\mathcal{P}_n)} \le \limsup_{n\to\infty}\frac{1}{\sqrt[n]{\phi_n(b\cdot\delta)}} = \limsup_{n\to\infty}\sup_{z\in\partial E_{b\cdot\delta}}\frac{1}{\sqrt[n]{\Phi_n(z)}} = \limsup_{n\to\infty}\frac{1}{\sqrt[n]{\Phi_n(z_n)}}$$

for some sequence $\{z_n\}_{n\in\mathbb{N}}\subset\partial E_{b\cdot\delta}$. We select a subsequence realizing the last supremum such that $z_n\to z_0\in\partial E_{b\cdot\delta}$ and then we see that

$$\limsup_{n\to\infty}\frac{1}{\sqrt[n]{\Phi_n(z_n)}} = \limsup_{n\to\infty}\frac{1}{\sqrt[n]{\Phi_n(z_0)}} = \frac{1}{\Phi_E(z_0)} \le \sup_{z\in\partial E_{b\cdot\delta}}\frac{1}{\Phi_E(z)} = \frac{1}{\phi_E(b\cdot\delta)}.$$

Finally we take the limit for $b\to 1$ and note that the function ϕ_E is continuous or equal to $+\infty$. $\square$

Definition 8.18. For a compact set $E\subset\subset\mathbb{C}$ and $\rho\ge 1$ we denote the level sets of the extremal function as follows:

$$C(E,\rho) := \{z\in\mathbb{C} \ :\ \Phi_E(z)=\rho\} = \{z\in\mathbb{C}\ :\ g_E(z)=\log\rho\}.$$

Note that for convenience we extend Green's function to the entire complex plane by putting $g_E(z):=0$ for all $z\in\hat{E}$.

Remark 8.19 [Gaier, chapter II §3.A theorem 1]. A more precise version of the previous corollary is obviously well known. If the set $E\subset\subset\mathbb{C}$ is not polar and for $f\in\mathcal{H}^\infty(E)$ we have

$$\rho := \sup\left\{\varrho \ :\ \exists\widetilde{f}\in\mathcal{H}^\infty\left(\widehat{C(E,\varrho)}\right) \text{ such that } \widetilde{f}_{|E}=f_{|E}\right\} > 1,$$

then

$$\limsup_{n\to\infty}\sqrt[n]{\operatorname{dist}_E(f,\mathcal{P}_n)} = \frac{1}{\rho}.$$

Corollary 8.20. *For any compact set $E\subset\subset\mathbb{C}$ we have $\mathcal{H}^\infty(E)_{|E}\subset s(E)$ if and only if it is polynomially convex.*

Proof. The implication $(\Longrightarrow)$ is the subject of remark 8.4.

Conversely, if the set E is polynomially convex and $f\in\mathcal{H}^\infty(E)$ then there exists a $\delta>0$ such that $f\in\mathcal{H}^\infty(E_\delta)$. From definitions 1.10 and 8.13 it follows that $\phi_E(\delta)>1$ and therefore by corollary 8.17 we have

$$\limsup_{n\to\infty}\sqrt[n]{\operatorname{dist}_E(f,\mathcal{P}_n)} < 1,$$

which implies that $f_{|E}\in s(E)$. $\square$

LEMMA 8.21. *For any L-regular compact set $E \subset\subset \mathbb{C}$, $\zeta \in E_1 \setminus \hat{E}$, $1 < \rho \leq \Phi_E(\zeta)$ and $n \in \mathbb{Z}_+$ we have*

$$\operatorname{dist}_E(f_\zeta, \mathcal{P}_n) \leq \frac{c \cdot (n+1)}{\operatorname{dist}\big(C(E,\rho), E\big) \cdot \operatorname{dist}(\zeta, E)} \cdot \left(\frac{\rho}{\Phi_E(\zeta)}\right)^{n+1},$$

where $c \geq 1$ depends only on the set E.

PROOF. We put

$$d := \max_{z \in C\left(E, \|\Phi_E\|_{E_1}\right)} \operatorname{dist}(z, E)$$

and $c := 1 + d + \operatorname{diam} E$, which depend only on the set E. Fix $\zeta \in E_1 \setminus \hat{E}$, $1 < \rho \leq \Phi_E(\zeta)$, $n \in \mathbb{Z}_+$ and consider any $\eta \in E_d \setminus E$. Note that for the Lagrange interpolation polynomial with knots in $n+1$ Fekete extremal points $\left\{z_j^{(n)}\right\}_{j=0,\dots,n} \subset E$ and $\omega_n(z) := \prod_{j=0}^n \left(z - z_j^{(n)}\right)$ we have

$$L_n f_\eta(z) = \frac{\omega_n(\eta) - \omega_n(z)}{\omega_n(\eta) \cdot (\eta - z)}.$$

This is true because clearly $L_n \in \mathcal{P}_n$ and for $j = 0, ..., n$ we have $\omega_n\left(z_j^{(n)}\right) = 0$ and hence

$$L_n f_\eta\left(z_j^{(n)}\right) = \frac{1}{\eta - z_j^{(n)}} = f_\eta\left(z_j^{(n)}\right).$$

Consequently, applying the properties of Lagrange interpolation polynomials discussed in remark 1.8, we see that for all $z \in E$ we have

$$\left|\frac{\omega_n(z)}{\omega_n(\eta)}\right| = |1 - (\eta - z) \cdot L_n f_\eta(z)| \leq 1 + \big(\operatorname{dist}(\eta, E) + \operatorname{diam} E\big) \cdot (n+1) \cdot ||f_\eta||_E \leq$$

$$\leq 1 + (d + \operatorname{diam} E) \cdot (n+1) \cdot \frac{1}{\operatorname{dist}(\eta, E)} \leq \frac{c \cdot (n+1)}{\operatorname{dist}(\eta, E)}.$$

Now put $h_n(\theta) := \log|\omega_n(\theta)| - (n+1) \cdot g_E(\theta)$, which is a harmonic function on $\mathbb{C} \setminus \hat{E}$, bounded when $\theta \to \infty$. For any $\theta \in C(E, \rho) \subset E_d \setminus \hat{E}$ and $z \in E$ we have

$$|\omega_n(\theta)| \geq \frac{\operatorname{dist}(\theta, E) \cdot |\omega_n(z)|}{c \cdot (n+1)} \geq \frac{\operatorname{dist}\big(C(E,\rho), E\big) \cdot |\omega_n(z)|}{c \cdot (n+1)},$$

$$h_n(\theta) \geq \log\left(\frac{\operatorname{dist}\big(C(E,\rho), E\big) \cdot |\omega_n(z)|}{c \cdot (n+1)}\right) - (n+1) \cdot \log \rho.$$

The L-regularity of the set E implies that the level set $C(E, \rho)$ is the boundary of the open domain $\Omega := \{z \in \mathbb{C} \ : \ \Phi_E(z) > \rho\}$ and $\zeta \in \bar{\Omega}$. Therefore the minimum principle for harmonic functions leads us to

$$h_n(\zeta) \geq \log\left(\frac{\operatorname{dist}\big(C(E,\rho), E\big) \cdot |\omega_n(z)|}{c \cdot (n+1)}\right) - (n+1) \cdot \log \rho$$

and this then implies that

$$\log|\omega_n(\zeta)| \geq \log\left(\frac{\operatorname{dist}\big(C(E,\rho), E\big) \cdot |\omega_n(z)|}{c \cdot (n+1)}\right) + (n+1) \cdot \big(g_E(\zeta) - \log \rho\big),$$

$$\left|\frac{\omega_n(z)}{\omega_n(\zeta)}\right| \leq \frac{c \cdot (n+1)}{\operatorname{dist}\big(C(E,\rho), E\big)} \cdot \left(\frac{\rho}{\Phi_E(\zeta)}\right)^{n+1}.$$

Returning to the Lagrange interpolation polynomial we obtain for all $z \in E$

$$|f_\zeta(z) - L_n f_\zeta(z)| = \left|\frac{\omega_n(z)}{\omega_n(\zeta) \cdot (\zeta - z)}\right| \leq \frac{c \cdot (n+1)}{\operatorname{dist}\big(C(E,\rho), E\big)} \cdot \left(\frac{\rho}{\Phi_E(\zeta)}\right)^{n+1} \cdot \frac{1}{|\zeta - z|}$$

and ultimately

$$\operatorname{dist}_E(f_\zeta, \mathcal{P}_n) \leq ||f_\zeta - L_n f_\zeta||_E \leq \frac{c \cdot (n+1)}{\operatorname{dist}\big(C(E,\rho), E\big) \cdot \operatorname{dist}(\zeta, E)} \cdot \left(\frac{\rho}{\Phi_E(\zeta)}\right)^{n+1}. \quad \square$$

LEMMA 8.22. *Assume that a polynomially convex compact set $E \subset\subset \mathbb{C}$ admits* $Ł S(s)$ *and* HCP(k) *for some $s, k \geq 1$, i.e. there exist $a_1, a_2 \geq 1$ such that for all $z \in E_1$*

$$\frac{1}{a_1} \cdot \operatorname{dist}(z, E)^s \leq g_E(z) \leq a_2 \cdot \operatorname{dist}(z, E)^{1/k}.$$

Then there exist $c_0, c_1 \geq 1$ dependent only on the set E such that

$$\forall \ell \geq 1 \quad \forall 0 < t \leq 1 \quad \sup_{n \in \mathbb{N}} \frac{n^\ell}{\phi_n(t)} \leq \left(\frac{c_1 \cdot \ell}{t^s}\right)^{\ell + c_0}.$$

PROOF. Fix $\ell \geq 1$ and $0 < t \leq 1$. By lemma 8.21 for arbitrary $\zeta \in \partial E_t$, $\rho := \sqrt{\Phi_E(\zeta)} = e^{g_E(\zeta)/2} > 1$ and $n \in \mathbb{N}$ we have

$$\operatorname{dist}_E(f_\zeta, \mathcal{P}_{n-1}) \leq \frac{c \cdot n}{\operatorname{dist}\big(C(E, \rho), E\big) \cdot t} \cdot \left(\frac{\rho}{\Phi_E(\zeta)}\right)^n,$$

where $c \geq 1$ depends only on the set E. We combine this with the result of lemma 8.12 to obtain

$$\frac{n^\ell}{\Phi_n(\zeta)} \leq \big(\operatorname{dist}(\zeta, E) + \operatorname{diam} E\big) \cdot \operatorname{dist}_E(f_\zeta, \mathcal{P}_{n-1}) \cdot n^\ell \leq \frac{\widetilde{c}}{\operatorname{dist}\big(C(E, \rho), E\big) \cdot t} \cdot n^{\ell+1} \cdot \left(\frac{1}{\rho}\right)^n \leq$$

$$\leq \frac{\widetilde{c}}{\operatorname{dist}\big(C(E, \rho), E\big) \cdot t} \cdot \left(\frac{\ell + 1}{e \cdot \log \rho}\right)^{\ell+1} \leq \frac{\widetilde{c}}{\operatorname{dist}\big(C(E, \rho), E\big) \cdot t} \cdot \left(\frac{2\ell}{g_E(\zeta)}\right)^{\ell+1},$$

where $\widetilde{c} := (1 + \operatorname{diam} E) \cdot c$, because for $a, b > 0$

$$\sup_{n \in \mathbb{R}_+} n^a \cdot e^{-b \cdot n} = \left(\frac{a}{b \cdot e}\right)^a.$$

By the assumption HCP(k) we know that for all $z \in C(E, \rho)$ we have $\log \rho = g_E(z) \leq a_2 \cdot \operatorname{dist}(z, E)^{1/k}$ and therefore

$$\frac{1}{\operatorname{dist}\big(C(E, \rho), E\big)} \leq \left(\frac{a_2}{\log \rho}\right)^k = \left(\frac{2a_2}{g_E(\zeta)}\right)^k.$$

On the other hand the assumption $ŁS(s)$ tells us that

$$\frac{t^s}{a_1} = \frac{1}{a_1} \cdot \operatorname{dist}(\zeta, E)^s \leq g_E(\zeta),$$

so we can combine these estimates to obtain

$$\frac{n^\ell}{\Phi_n(\zeta)} \leq \frac{\widetilde{c}}{t} \cdot \left(\frac{2a_2}{g_E(\zeta)}\right)^k \cdot \left(\frac{2\ell}{g_E(\zeta)}\right)^{\ell+1} \leq \frac{\widetilde{c}}{t} \cdot \left(\frac{2a_1 \cdot a_2}{t^s}\right)^k \cdot \left(\frac{2a_1 \cdot \ell}{t^s}\right)^{\ell+1} \leq \left(\frac{c_1 \cdot \ell}{t^s}\right)^{\ell + c_0},$$

where $c_0 := k + 2$ and $c_1 := \widetilde{c} \cdot 2a_1 \cdot a_2$ depend only on the set E.

Finally we conclude that

$$\sup_{n \in \mathbb{N}} \frac{n^\ell}{\phi_n(t)} = \sup_{n \in \mathbb{N}} \sup_{\zeta \in \partial E_t} \frac{n^\ell}{\Phi_n(\zeta)} \leq \left(\frac{c_1 \cdot \ell}{t^s}\right)^{\ell + c_0}. \quad \square$$

PROPOSITION 8.23. *For any compact set $E \subset\subset \mathbb{C}$ and $s, v \geq 1$ the following two conditions are equivalent:*

$$\text{WJP}(s) \quad \textit{i.e.} \quad \exists c_0 \geq 0 \quad \forall \ell \geq 1 \quad \exists c_\ell \geq 1 \quad \forall 0 < \delta \leq 1 \quad \forall f \in \mathcal{H}^\infty(E_\delta) \quad :$$

(i) $$|f_{|E}|_\ell \leq \left(\frac{c_\ell}{\delta^s}\right)^{\ell + c_0} \cdot ||f||_{E_\delta},$$

(ii) $$\exists \widetilde{c}_0 \geq 0 \quad \forall \ell \geq 1 \quad \exists \widetilde{c}_\ell \geq 1 \quad \forall 0 < t \leq 1 \quad \forall n \in \mathbb{N} \quad : \quad \frac{n^\ell}{\phi_{n+1}(t)} \leq \left(\frac{\widetilde{c}_\ell}{t^s}\right)^{\ell+\widetilde{c}_0}.$$

Furthermore we have $c_\ell \leq c_1 \cdot \ell^v$ for all $\ell \geq 1$, i.e. the set E admits JP(s,v), *if and only if $\widetilde{c}_\ell \leq \widetilde{c}_1 \cdot \ell^v$ for all $\ell \geq 1$.*

PROOF. (i)$\Longrightarrow$(ii) Fix $0 < t \leq 1$ and arbitrary $\zeta \in \partial E_t$. Let $\delta := \frac{t}{2}$ and $f_\zeta \in \mathcal{H}^\infty(E_\delta)$ as in definition 8.3. Then by the assumption we have for all $\ell \geq 1$ and $n \in \mathbb{N}$

$$n^\ell \cdot \operatorname{dist}_E(f_\zeta, \mathcal{P}_n) \leq |f_{|E}|_\ell \leq \left(\frac{c_\ell}{\delta^s}\right)^{\ell+c_0} \cdot ||f_\zeta||_{E_\delta} = \left(\frac{2^s \cdot c_\ell}{t^s}\right)^{\ell+c_0} \cdot \frac{2}{t} \leq \left(\frac{2^s \cdot c_\ell}{t^s}\right)^{\ell+c_0+1}.$$

Lemma 8.12 implies that

$$\frac{n^\ell}{\Phi_{n+1}(\zeta)} \leq \big(\operatorname{dist}(\zeta, E) + \operatorname{diam} E\big) \cdot n^\ell \cdot \operatorname{dist}_E(f_\zeta, \mathcal{P}_n) \leq \left(\frac{\widetilde{c}_\ell}{t^s}\right)^{\ell+\widetilde{c}_0},$$

where $\widetilde{c}_0 := c_0 + 1$ and $\widetilde{c}_\ell := (1 + \operatorname{diam} E) \cdot 2^s \cdot c_\ell$ depend only on the set E. Therefore, since $\zeta \in \partial E_t$ was arbitrary, we conclude that

$$\frac{n^\ell}{\phi_{n+1}(t)} = \sup_{\zeta \in \partial E_t} \frac{n^\ell}{\Phi_{n+1}(\zeta)} \leq \left(\frac{\widetilde{c}_\ell}{t^s}\right)^{\ell+\widetilde{c}_0}.$$

Furthermore if $c_\ell \leq c_1 \cdot \ell^v$ for all $\ell \geq 1$, then we have also $\widetilde{c}_\ell \leq (1 + \operatorname{diam} E) \cdot 2^s \cdot c_1 \cdot \ell^v = \widetilde{c}_1 \cdot \ell^v$.

(i)$\Longleftarrow$(ii) Fix $\ell \geq 1$, $0 < \delta \leq 1$ and $f \in \mathcal{H}^\infty(E_\delta)$. We now apply proposition 8.16 with $b := \frac{1}{2}$ and $t := b \cdot \delta$ to obtain for any $n \in \mathbb{N}$

$$\begin{aligned} n^\ell \cdot \operatorname{dist}_E(f, \mathcal{P}_n) &\leq \frac{c \cdot n^\ell \cdot \|f\|_{E_\delta}}{(1-b) \cdot \delta^2 \cdot \phi_{n+1}(b \cdot \delta)} = \frac{2c}{\delta^2} \cdot \frac{n^\ell}{\phi_{n+1}(t)} \cdot \|f\|_{E_\delta} \leq \frac{2c}{\delta^2} \cdot \left(\frac{\widetilde{c}_\ell}{t^s}\right)^{\ell+\widetilde{c}_0} \cdot \|f\|_{E_\delta} = \\ &= \frac{2c}{\delta^2} \cdot \left(\frac{2^s \cdot \widetilde{c}_\ell}{\delta^s}\right)^{\ell+\widetilde{c}_0} \cdot \|f\|_{E_\delta} \leq \left(\frac{c \cdot 2^s \cdot \widetilde{c}_\ell}{\delta^s}\right)^{\ell+\widetilde{c}_0+2} \cdot \|f\|_{E_\delta}. \end{aligned}$$

From this it follows that

$$|f_{|E}|_\ell = \|f\|_E + \sup_{n \in \mathbb{N}} n^\ell \cdot \operatorname{dist}_E(f, \mathcal{P}_n) \leq \left(\frac{c_\ell}{\delta^s}\right)^{\ell+c_0} \cdot ||f||_{E_\delta},$$

where $c_0 := \widetilde{c}_0 + 2$ and $c_\ell := 1 + c \cdot 2^s \cdot \widetilde{c}_\ell$ depend only on the set E. Furthermore if $\widetilde{c}_\ell \leq \widetilde{c}_1 \cdot \ell^v$ for all $\ell \geq 1$, then we have also $c_\ell \leq 1 + c \cdot 2^s \cdot \widetilde{c}_1 \cdot \ell^v \leq c_1 \cdot \ell^v$. $\square$

THEOREM 8.24 JACKSON'S THEOREM IN THE COMPLEX PLANE. *Any polynomially convex compact set $E \subset\subset \mathbb{C}$ admitting* ŁS(s) *and* HCP, *where $s \geq 1$, admits* JP$(s,1)$.

PROOF. This is an immediate consequence of lemma 8.22, remark 8.14 and proposition 8.23. $\square$

REMARK 8.25. In [Newman, lemma 4] an estimate equivalent to JP$(1,1)$ with $c_0 = 2$ was proven for all simply connected bounded regions with boundaries that are Jordan curves of class $\mathcal{C}^{1+\delta}$.

PROPOSITION 8.26. *For any compact set $E \subset\subset \mathbb{C}$ and $s, s', v \geq 1$ such that $s' > s$ we have*

$$\mathrm{JP}(s,v) \implies \mathrm{WJP}(s) \implies \text{ŁS}(s'),$$

$$\mathrm{JP}(s,1) \implies \text{ŁS}(s).$$

PROOF. The implication JP$(s,v) \Longrightarrow$ WJP(s) follows straight from definition 8.2.

If we assume that the set E admits WJP(s) then proposition 8.23 implies that

$$\exists \widetilde{c}_0 \geq 0 \quad \forall \ell \geq 1 \quad \exists \widetilde{c}_\ell \geq 1 \quad \forall 0 < t \leq 1 \quad \forall n \in \mathbb{N} \quad : \quad \frac{n^\ell}{\phi_{n+1}(t)} \leq \left(\frac{\widetilde{c}_\ell}{t^s}\right)^{\ell+\widetilde{c}_0}.$$

From this it follows that for arbitrary $0 < t \le 1$, $\zeta \in \partial E_t$, $n \in \mathbb{N}$ and $\ell \ge 1$ we have

$$g_E(\zeta) = \log \Phi_E(\zeta) \ge \log \sqrt[n+1]{\Phi_{n+1}(\zeta)} \ge \log \sqrt[n+1]{\phi_{n+1}(t)} \ge \frac{1}{n+1} \cdot \log \left(n^\ell \cdot \left(\frac{t^s}{\widetilde{c}_\ell} \right)^{\ell + \widetilde{c}_0} \right).$$

Specifically by taking $n \in \mathbb{N}$ such that

$$e \cdot \left(\frac{\widetilde{c}_\ell}{t^s} \right)^{1+\widetilde{c}_0/\ell} \le n < e \cdot \left(\frac{\widetilde{c}_\ell}{t^s} \right)^{1+\widetilde{c}_0/\ell} + 1$$

we find that

$$g_E(\zeta) \ge \frac{1}{n+1} \cdot \log e^\ell \ge \frac{\ell}{e \cdot \left(\frac{\widetilde{c}_\ell}{t^s} \right)^{1+\widetilde{c}_0/\ell} + 2} \ge$$

$$\ge \frac{\ell}{e \cdot \widetilde{c}_\ell^{1+\widetilde{c}_0/\ell} + 2} \cdot t^{s \cdot (1+\widetilde{c}_0/\ell)} = \frac{\ell}{e \cdot \widetilde{c}_\ell^{1+\widetilde{c}_0/\ell} + 2} \cdot \operatorname{dist}(\zeta, E)^{s \cdot (1+\widetilde{c}_0/\ell)}$$

and by taking ℓ sufficiently large, we obtain ŁS(s') for any $s' > s$.

Finally if we assume additionally that $\widetilde{c}_\ell \le \widetilde{c}_1 \cdot \ell$, i.e. the set E admits JP$(s,1)$, then we can take the limit of the last estimate for $\ell \to \infty$ to obtain

$$g_E(\zeta) \ge \lim_{\ell\to\infty} \frac{\ell}{e \cdot \widetilde{c}_\ell^{1+\widetilde{c}_0/\ell} + 2} \cdot \operatorname{dist}(\zeta, E)^{s \cdot (1+\widetilde{c}_0/\ell)} \ge \lim_{\ell\to\infty} \frac{\ell}{e \cdot (\widetilde{c}_1 \cdot \ell)^{1+\widetilde{c}_0/\ell} + 2} \cdot \operatorname{dist}(\zeta, E)^{s \cdot (1+\widetilde{c}_0/\ell)} =$$

$$= \lim_{\ell\to\infty} \frac{1}{e \cdot \widetilde{c}_1 \cdot (\widetilde{c}_1 \cdot \ell)^{\widetilde{c}_0/\ell} + 2/\ell} \cdot \operatorname{dist}(\zeta, E)^{s \cdot (1+\widetilde{c}_0/\ell)} = \frac{1}{e \cdot \widetilde{c}_1} \cdot \operatorname{dist}(\zeta, E)^s. \quad \square$$

PROPOSITION 8.27. *Any compact set $E \subset\subset \mathbb{R}$ admits ŁS(1).*

PROOF. Fix $z \in E_1 \setminus E$ and write $z = x + y \cdot i$ where $x, y \in \mathbb{R}$. We denote

$$a := \min_{\zeta \in E} |\zeta - x|, \qquad b := 1 + \max_{\zeta \in E} |\zeta - x| > a,$$

$$c := 2 + \operatorname{diam} E \ge b, \qquad d := \operatorname{dist}(z, E) = \sqrt{a^2 + y^2} \le 1 < c,$$

and observe that c is independent of the choice of z. Consider the mapping

$$\psi : \mathbb{C} \to \mathbb{C} \qquad \psi(\zeta) := \frac{(b^2 + a^2)/2 - (\zeta - x)^2}{(b^2 - a^2)/2}$$

and note that $\psi(E) \subset I := [-1, +1]$ but

$$\psi(z) = \frac{(b^2 + a^2)/2 - (y \cdot i)^2}{(b^2 - a^2)/2} = \frac{b^2 + a^2 + 2y^2}{b^2 - a^2} = 1 + 2 \cdot \frac{a^2 + y^2}{b^2 - a^2} = 1 + 2 \cdot \frac{d^2}{b^2 - a^2} > 1.$$

Now if $p \in \mathcal{P}_n$ with $n \in \mathbb{N}$ and $||p||_I \le 1$ then $p \circ \psi \in \mathcal{P}_{2n}$ and $||p \circ \psi||_E \le 1$. Therefore

$$\Phi_E(z) = \sup \left\{ \sqrt[n]{|p(z)|} \ : \ n \in \mathbb{N},\ p \in \mathcal{P}_n,\ ||p||_E \le 1 \right\} \ge$$

$$\ge \sup \left\{ \sqrt[2n]{|p \circ \psi(z)|} \ : \ n \in \mathbb{N},\ p \in \mathcal{P}_n,\ ||p||_I \le 1 \right\} = \sqrt{\Phi_I\big(\psi(z)\big)}.$$

Because of the fact that $\Psi(\zeta) := \frac{1}{2} \cdot \left(\zeta + \frac{1}{\zeta} \right)$ is a conformal mapping of the exterior of the unit ball onto the exterior of the line segment I, theorem 1.11.c leads us to

$$\Phi_I(\zeta) = |\Psi^{-1}(\zeta)| = \zeta + \sqrt{\zeta^2 - 1}$$

for all $\zeta \in \mathbb{R}$ such that $\zeta \ge 1$. Finally we see that

$$\Phi_E(z) \ge \sqrt{\Phi_I\big(\psi(z)\big)} = \sqrt{\psi(z) + \sqrt{\psi(z)^2 - 1}} =$$

$$= \sqrt{1 + 2 \cdot \frac{d^2}{b^2 - a^2} + \sqrt{\left(1 + 2 \cdot \frac{d^2}{b^2 - a^2} \right)^2 - 1}} \ge \sqrt{1 + \sqrt{\left(1 + 2 \cdot \frac{d^2}{c^2} \right)^2 - 1}} =$$

$$= \sqrt{1 + \sqrt{4 \cdot \frac{d^2}{c^2} + 4 \cdot \frac{d^4}{c^4}}} \ge \sqrt{1 + \sqrt{4 \cdot \frac{d^2}{c^2}}} = \sqrt{1 + 2 \cdot \frac{d}{c}} \ge 1 + \frac{2d}{3c} = 1 + \frac{2}{3c} \cdot \operatorname{dist}(z, E)$$

because $0 < \frac{2d}{c} < 2$ and $\sqrt{1+t} \ge 1 + \frac{t}{3}$ for $0 \le t \le 3$. $\square$

Corollary 8.28. *Every compact set* $E \subset\subset \mathbb{R} \subset \mathbb{C}$*, which admits* HCP*, also admits* JP$(1,1)$.

Proof. This is an immediate consequence of proposition 8.27 and theorem 8.24. $\square$

Note that this corollary improves the result of corollary 8.7.

Lemma 8.29 [cf. Pleśniak 6, theorem 1]. *Assume that the compact set* $E \subset\subset \mathbb{C}$ *is the sum of two polynomially convex, disjoint, non-polar compact sets, i.e.* $E = A \cup B$, $A = \hat{A}$, $B = \hat{B}$, $A \cap B = \emptyset$, $\operatorname{cap} A > 0$ *and* $\operatorname{cap} B > 0$. *Then for any function* $f \in \mathcal{C}(E)$ *such that* $f_{|A} \in s(A)$ *and* $f_{|B} \in s(B)$*, we have* $f \in s(E)$ *and furthermore we can estimate its Jackson norms on the set* E *by its Jackson norms on the sets* A *and* B *as follows:*

$$\forall \ell \geq 1 \quad : \quad |f|_\ell \leq (c \cdot \ell)^\ell \cdot \left(|f_{|A}|_\ell + |f_{|B}|_\ell\right),$$

where the constant $c \geq 1$ *depends only on the sets* A *and* B*. Note that these are three different Jackson norms and only the domain of the function indicates which norm is meant.*

Proof. We put

$$\chi(z) := \begin{cases} 0 & \text{if } z \in A \\ 1 & \text{if } z \in B \end{cases}$$

and we note that this function can be extended holomorphically so that $\chi \in \mathcal{H}^\infty\left(\widehat{C(E,\rho)}\right)$ for some $\rho > 1$. Therefore by remark 8.19 we can find a constant $M \geq 1$ such that

$$\forall n \in \mathbb{N} \quad : \quad \operatorname{dist}_E(\chi, \mathcal{P}_n) \leq \frac{M}{\rho^n}.$$

Let also $x := \max\{||\Phi_A||_B, ||\Phi_B||_A\}$ and note that $1 < x < +\infty$, because the sets A and B are non-polar and compact. Therefore we can determine a number $a \in \mathbb{N}$ so that $t := \frac{\rho^a}{x} > 1$.

Now fix an arbitrary function $f \in \mathcal{C}(E)$, such that $f_{|A} \in s(A)$ and $f_{|B} \in s(B)$, and $\ell \geq 1$. Find three sequences of polynomials of best approximation for the functions $f_{|A}$, $f_{|B}$ and χ on the sets A, B and E respectively, i.e. $p_n, q_n, r_n \in \mathcal{P}_n$, $\|f - p_n\|_A = \operatorname{dist}_A(f, \mathcal{P}_n)$, $\|f - q_n\|_B = \operatorname{dist}_B(f, \mathcal{P}_n)$ and $\|\chi - r_n\|_E = \operatorname{dist}_E(\chi, \mathcal{P}_n)$ for each $n \in \mathbb{Z}_+$. Using the Bernstein-Walsh-Siciak inequality we see that

$$\|p_n\|_A = \|f - (f - p_n)\|_A \leq \|f\|_A + \|f - p_n\|_A = \|f\|_A + \operatorname{dist}_A(f, \mathcal{P}_n) \leq 2 \cdot \|f\|_A,$$

$$\|p_n\|_B \leq \|\Phi_A\|_B^n \cdot \|p_n\|_A \leq x^n \cdot \|p_n\|_A \leq 2x^n \cdot \|f\|_A$$

and similarly

$$\|q_n\|_B \leq 2 \cdot \|f\|_B,$$
$$\|q_n\|_A \leq 2x^n \cdot \|f\|_B.$$

For each $n \in \mathbb{Z}_+$ we put

$$s_n(z) := p_n(z) \cdot \left(1 - r_{a \cdot n}(z)\right) + q_n(z) \cdot r_{a \cdot n}(z)$$

so that $s_n \in \mathcal{P}_{(a+1)\cdot n}$. This way we obtain

$$\|f - s_n\|_A = \|f - p_n + r_{a\cdot n} \cdot (p_n - q_n)\|_A \leq \|f - p_n\|_A + \|r_{a\cdot n}\|_A \cdot \|p_n - q_n\|_A \leq$$
$$\leq \operatorname{dist}_A(f, \mathcal{P}_n) + \operatorname{dist}_E(\chi, \mathcal{P}_{a\cdot n}) \cdot \left(\|p_n\|_A + \|q_n\|_A\right) \leq \frac{|f_{|A}|_\ell}{n^\ell} + \frac{M}{\rho^{a\cdot n}} \cdot \left(2 \cdot \|f\|_A + 2x^n \cdot \|f\|_B\right) \leq$$
$$\leq \frac{|f_{|A}|_\ell}{n^\ell} + \frac{M}{\rho^{a\cdot n}} \cdot 4x^n \cdot \|f\|_E = \frac{|f_{|A}|_\ell}{n^\ell} + \frac{4M}{t^n} \cdot \|f\|_E,$$

$$\|f - s_n\|_B = \|f - q_n + (1 - r_{a\cdot n}) \cdot (q_n - p_n)\|_B \leq \|f - q_n\|_B + \|1 - r_{a\cdot n}\|_B \cdot \|q_n - p_n\|_B \leq$$
$$\leq \operatorname{dist}_B(f, \mathcal{P}_n) + \operatorname{dist}_E(\chi, \mathcal{P}_{a\cdot n}) \cdot \left(\|q_n\|_B + \|p_n\|_B\right) \leq \frac{|f_{|B}|_\ell}{n^\ell} + \frac{M}{\rho^{a\cdot n}} \cdot \left(2 \cdot \|f\|_B + 2x^n \cdot \|f\|_A\right) \leq$$
$$\leq \frac{|f_{|B}|_\ell}{n^\ell} + \frac{M}{\rho^{a\cdot n}} \cdot 4x^n \cdot \|f\|_E = \frac{|f_{|B}|_\ell}{n^\ell} + \frac{4M}{t^n} \cdot \|f\|_E,$$

which leads us to

$$n^\ell \cdot \mathrm{dist}_E(f, \mathcal{P}_{(a+1)\cdot n}) \le n^\ell \cdot \|f - s_n\|_E \le |f_{|A}|_\ell + |f_{|B}|_\ell + \frac{4M \cdot n^\ell}{t^n} \cdot \|f\|_E \le$$
$$\le |f_{|A}|_\ell + |f_{|B}|_\ell + 4M \cdot \left(\frac{\ell}{e \cdot \log t}\right)^\ell \cdot \|f\|_E \le (\widetilde{c} \cdot \ell)^\ell \cdot \left(|f_{|A}|_\ell + |f_{|B}|_\ell\right),$$

where $\widetilde{c} := 1 + \max\left\{\frac{4M}{e \cdot \log t}, 1\right\}$ depends on the sets A and B but not on the choice of the function f and ℓ. Finally for arbitrary $n \in \mathbb{Z}_+$ we can find $N \in \mathbb{Z}_+$ such that $(a+1) \cdot N \le n < (a+1) \cdot (N+1)$ to see that

$$n^\ell \cdot \mathrm{dist}_E(f, \mathcal{P}_n) \le \big((a+1) \cdot (N+1)\big)^\ell \cdot \mathrm{dist}_E(f, \mathcal{P}_{(a+1)\cdot N}) \le$$
$$\le (4a)^\ell \cdot N^\ell \cdot \mathrm{dist}_E(f, \mathcal{P}_{(a+1)\cdot N}) \le (4a \cdot \widetilde{c} \cdot \ell)^\ell \cdot \left(|f_{|A}|_\ell + |f_{|B}|_\ell\right),$$

$$|f|_\ell = \|f\|_E + \sup_{n \in \mathbb{N}} n^\ell \cdot \mathrm{dist}_E(f, \mathcal{P}_n) \le (c \cdot \ell)^\ell \cdot \left(|f_{|A}|_\ell + |f_{|B}|_\ell\right) < +\infty,$$

where $c := 1 + 4a \cdot \widetilde{c}$ also depends only on the sets A and B. □

PROPOSITION 8.30. *Assume that the compact set $E \subset\subset \mathbb{C}$ is the sum of two polynomially convex, disjoint compact sets, i.e. $E = A \cup B$, $A = \hat{A}$, $B = \hat{B}$ and $A \cap B = \emptyset$. If the set E admits* JP *or* WJP*, then both subsets A and B admit* JP*, respectively* WJP*, with the same coefficients.*

Conversely if both sets A and B are additionally non-polar and they both admit JP(s, v) *or* WJP(s)*, for some $s, v \ge 1$, then the set E admits* JP$(s, v+1)$*, respectively* WJP(s).

PROOF. In order to prove the first assertion, we first blow these sets up so that $\mathrm{dist}(A, B) > 2$ and next we apply proposition 8.23 to obtain condition (ii) for the set E. Then we note that the extremal functions Φ_n of the sets A and B are bounded below by the extremal functions of the set E and this way we obtain the same condition (ii) for the sets A and B. Finally we apply proposition 8.23 again to conclude that they too admit JP, respectively WJP, with the same coefficients.

The second assertion follows straight from lemma 8.29. □

CHAPTER IX

EXTENSION PROPERTY BY PLEŚNIAK (EXT)

Definition 9.1 [cf. Eggink, definition 3.3; cf. Pleśniak 1, theorem 3.3]. A compact set $E \subset\subset \mathbb{C}$ admits the Extension Property by Pleśniak EXT if it is $\mathcal{A}^\infty$-determining and there exists a continuous and linear extension operator $L : s(E) \longrightarrow \mathcal{A}^\infty(E)$, such that

$$\forall f \in s(E) \quad : \quad (Lf)_{|E} \equiv f.$$

Here the space $\mathcal{A}^\infty(E)$ is normed by the usual seminorms $\|D^\alpha f\|_K$, where $\alpha \in \mathbb{Z}_+^2$, $K \subset\subset \mathbb{C}$.

Building on earlier joint work with W. Pawłucki [Pawłucki-Pleśniak 1; Pawłucki-Pleśniak 2], W. Pleśniak originally proved the equivalence of GMI and EXT for $\mathcal{C}^\infty$-determining compact subsets of $\mathbb{R}^N$, $N \in \mathbb{N}$.

Theorem 9.2 [cf. Eggink, theorem 3.4; cf. Pleśniak 1, theorem 3.3]. *For any polynomially convex compact set $E \subset\subset \mathbb{C}$ we have*

$$\text{GMI} \iff \text{EXT}.$$

Proof. ($\Longrightarrow$) We assume that the set E admits GMI(k), i.e.

$$\exists M \geq 1 \quad \forall n \in \mathbb{N} \quad \forall p \in \mathcal{P}_n \quad : \quad \|p'\|_E \leq M \cdot n^k \cdot \|p\|_E.$$

Proposition 1.18 implies that

$$\forall n \in \mathbb{N} \quad \forall p \in \mathcal{P}_n \quad : \quad \|p\|_{E_{1/n^k}} \leq \widetilde{M} \cdot \|p\|_E,$$

where $\widetilde{M} := e^M$. From propositions 1.21 and 5.4 we know that the set E is $\mathcal{A}^\infty$-determining.

Now we fix a function $f \in s(E)$ and for each $n \in \mathbb{Z}_+$ we take $L_n f \in \mathcal{P}_n$ to be a Lagrange interpolation polynomial of this function with fixed knots in $n+1$ Fekete extremal points of the set E. This means that

$$L_n f(z) := \sum_{\mu=0}^{n} f\left(\zeta_\mu^{(n)}\right) \cdot L_{n,\mu}\left(z; \zeta_0^{(n)}, \ldots, \zeta_n^{(n)}\right),$$

where for $n \in \mathbb{N}$ and $\mu = 0, \ldots, n$ we put

$$L_{n,\mu}\left(z; \zeta_0^{(n)}, \ldots, \zeta_n^{(n)}\right) := \prod_{\substack{\nu=0,\ldots,n \\ \nu \neq \mu}} \frac{z - \zeta_\nu^{(n)}}{\zeta_\mu^{(n)} - \zeta_\nu^{(n)}},$$

where the set $\left\{\zeta_0^{(n)}, \ldots, \zeta_n^{(n)}\right\} \subset E$ of Fekete extremal points is chosen so that it realizes the maximum on $E^{n+1} = \overbrace{E \times \cdots \times E}^{n+1 \text{ times}}$ of the expression $\prod_{0 \leq \mu < \nu \leq n} \left|\zeta_\nu^{(n)} - \zeta_\mu^{(n)}\right|$, while $\zeta_0^{(0)}$ is an arbitrary point of the set E and $L_{0,0}\left(z; \zeta_0^{(0)}\right) := 1$. This implies that for all $n \in \mathbb{Z}_+$ and $\mu, \nu = 0, \ldots, n$ we have

$$L_{n,\mu}\left(\zeta_\nu^{(n)}; \zeta_0^{(n)}, \ldots, \zeta_n^{(n)}\right) = \begin{cases} 0 & \text{if } \mu \neq \nu, \\ 1 & \text{if } \mu = \nu, \end{cases}$$

and thus $L_n f\left(\zeta_\nu^{(n)}\right) = f\left(\zeta_\nu^{(n)}\right)$. Also from the choice of the Fekete extremal points it follows that for all $z \in E$ we have $\left|L_{n,\mu}\left(z;\zeta_0^{(n)},\dots,\zeta_n^{(n)}\right)\right| \le 1$ and therefore

$$\|L_n f\|_E \le \sum_{\mu=0}^{n} \|f\|_E \cdot \left\|L_{n,\mu}\left(\,\cdot\,;\zeta_0^{(n)},\dots,\zeta_n^{(n)}\right)\right\|_E = (n+1)\cdot\|f\|_E.$$

We define the desired extension Lf as follows:

$$Lf := u_1 \cdot L_0 f + \sum_{n=1}^{\infty} u_n \cdot (L_n f - L_{n-1} f),$$

where $u_n \in \mathcal{C}^\infty(\mathbb{C})$, $n \in \mathbb{N}$, is a sequence of cutoff functions as constructed in proposition 6.8 for the compact set $K := E$ and radii $\epsilon_n := \frac{1}{n^k}$.

We will show that this series is convergent together with all its derivatives. For this purpose let's fix $\alpha \in \mathbb{Z}_+^2$ and $n \in \mathbb{N}$. We use the Leibniz rule to see that

$$\begin{aligned}\|D^\alpha\big(u_n\cdot(L_nf-L_{n-1}f)\big)\|_{\mathbb{C}} &\le \sum_{\substack{\beta\in\mathbb{Z}_+^2\\ \beta\le\alpha}} \binom{\alpha}{\beta}\cdot\|D^\beta u_n\cdot D^{\alpha-\beta}(L_nf-L_{n-1}f)\|_{\mathbb{C}} =\\ &= \sum_{\beta\le\alpha}\binom{\alpha}{\beta}\cdot\|D^\beta u_n\cdot D^{\alpha-\beta}(L_nf-L_{n-1}f)\|_{E_{1/n^k}} \le \dots\end{aligned}$$

since $u_n \equiv 0$ outside of E_{1/n^k},

$$\begin{aligned}\dots &\le \sum_{\beta\le\alpha}\binom{\alpha}{\beta}\cdot\|D^\beta u_n\|_{E_{1/n^k}}\cdot\|D^{\alpha-\beta}(L_nf-L_{n-1}f)\|_{E_{1/n^k}} \le\\ &\le \sum_{\beta\le\alpha}\binom{\alpha}{\beta}\cdot C_{|\beta|}\cdot n^{k\cdot|\beta|}\cdot\widetilde{M}\cdot\|D^{\alpha-\beta}(L_nf-L_{n-1}f)\|_E \le \dots\end{aligned}$$

because of the properties of the cutoff functions u_n and the fact that $D^{\alpha-\beta}(L_nf-L_{n-1}f)$ is a holomorphic polynomial of degree n at most,

$$\begin{aligned}\dots &\le \sum_{\substack{\beta\le\alpha\\ \beta_2=\alpha_2}}\binom{\alpha}{\beta}\cdot C_{|\beta|}\cdot n^{k\cdot|\beta|}\cdot\widetilde{M}\cdot M^{\alpha_1-\beta_1}\cdot n^{k\cdot(\alpha_1-\beta_1)}\cdot\|L_nf-L_{n-1}f\|_E =\\ &= \sum_{\substack{\beta\le\alpha\\ \beta_2=\alpha_2}}\binom{\alpha}{\beta}\cdot C_{|\beta|}\cdot\widetilde{M}\cdot M^{\alpha_1-\beta_1}\cdot n^{k\cdot|\alpha|}\cdot\|L_nf-L_{n-1}f\|_E = \widetilde{C}_\alpha\cdot n^{k\cdot|\alpha|}\cdot\|L_nf-L_{n-1}f\|_E,\end{aligned}$$

where $\widetilde{C}_\alpha := \sum_{\beta_1\le\alpha_1}\binom{\alpha_1}{\beta_1}\cdot C_{\beta_1+\alpha_2}\cdot\widetilde{M}\cdot M^{\alpha_1-\beta_1}$ are constants depending solely on the set E.

Now take $p_n \in \mathcal{P}_n$ to be any polynomial of best approximation, i.e. $\|f-p_n\|_E = \operatorname{dist}_E(f,\mathcal{P}_n)$. Since L_n is a linear operator preserving polynomials of degree n or less, we see that

$$\|L_nf-p_n\|_E = \|L_nf-L_n(p_{n|E})\|_E = \|L_n(f-p_{n|E})\|_E \le (n+1)\cdot\|f-p_n\|_E = (n+1)\cdot\operatorname{dist}_E(f,\mathcal{P}_n),$$

$$\|f-L_nf\|_E = \|(f-p_n)-(L_nf-p_n)\|_E \le \|f-p_n\|_E + \|L_nf-p_n\|_E \le (n+2)\cdot\operatorname{dist}_E(f,\mathcal{P}_n).$$

Furthermore we have

$$\begin{aligned}\|L_nf-L_{n-1}f\|_E &= \|(f-L_nf)-(f-L_{n-1}f)\|_E \le \|f-L_nf\|_E + \|f-L_{n-1}f\|_E \le\\ &\le (n+2)\cdot\operatorname{dist}_E(f,\mathcal{P}_n) + (n+1)\cdot\operatorname{dist}_E(f,\mathcal{P}_{n-1}) \le (2n+3)\cdot\operatorname{dist}_E(f,\mathcal{P}_{n-1}),\end{aligned}$$

and therefore

$$\|D^\alpha\big(u_n \cdot (L_n f - L_{n-1} f)\big)\|_{\mathbb{C}} \leq \widetilde{C}_\alpha \cdot n^{k\cdot|\alpha|} \cdot (2n+3) \cdot \operatorname{dist}_E(f, \mathcal{P}_{n-1}),$$

which for all $n \geq 2$ leads us to

$$\|D^\alpha\big(u_n \cdot (L_n f - L_{n-1} f)\big)\|_{\mathbb{C}} \leq \widetilde{C}_\alpha \cdot \frac{n^{k\cdot|\alpha|} \cdot (2n+3)}{(n-1)^{k\cdot|\alpha|+3}} \cdot |f|_{k\cdot|\alpha|+3}.$$

Finally we obtain

$$\begin{aligned}\|D^\alpha Lf\|_{\mathbb{C}} &\leq \|D^\alpha(u_1 \cdot L_0 f)\|_{\mathbb{C}} + \sum_{n=1}^{\infty} \|D^\alpha\big(u_n \cdot (L_n f - L_{n-1} f)\big)\|_{\mathbb{C}} \leq \\ &\leq C_{|\alpha|} \cdot |f(\zeta_0^{(0)})| + 5\widetilde{C}_\alpha \cdot \operatorname{dist}_E(f, \mathcal{P}_0) + \widetilde{C}_\alpha \cdot S_{|\alpha|} \cdot |f|_{k\cdot|\alpha|+3} \leq \\ &\leq \big(C_{|\alpha|} + 5\widetilde{C}_\alpha + \widetilde{C}_\alpha \cdot S_{|\alpha|}\big) \cdot |f|_{k\cdot|\alpha|+3} < +\infty,\end{aligned}$$

where for $t \in \mathbb{Z}_+$ we put

$$S_t := \sum_{n=2}^{\infty} \frac{n^{k\cdot t} \cdot (2n+3)}{(n-1)^{k\cdot t+3}} \leq \sum_{n=2}^{\infty} \frac{2^{k\cdot t} \cdot (n-1)^{k\cdot t} \cdot 7 \cdot (n-1)}{(n-1)^{k\cdot t+3}} = 7 \cdot 2^{k\cdot t} \cdot \sum_{n=2}^{\infty} \frac{1}{(n-1)^2} = \frac{7\pi^2}{6} \cdot 2^{k\cdot t} < +\infty.$$

As the constants $\big(C_{|\alpha|} + 5\widetilde{C}_\alpha + \widetilde{C}_\alpha \cdot S_{|\alpha|}\big)$ depend only on the set E, this proves the continuity of the operator $L : s(E) \to \mathcal{C}^\infty(\mathbb{C})$, the linearity of which is obvious.

Now we know that $Lf \in \mathcal{C}^\infty(\mathbb{C})$ we will show that $(Lf)_{|E} \equiv f$ and $Lf \in \mathcal{A}^\infty(E)$. For this purpose let's fix $z \in E$ and $\alpha \in \mathbb{Z}_+^2$ such that $\alpha_2 \geq 1$. It is easily seen that

$$\begin{aligned}Lf(z) &= u_1(z) \cdot L_0 f(z) + \sum_{n=1}^{\infty} u_n(z) \cdot \big(L_n f(z) - L_{n-1} f(z)\big) = \\ &= L_0 f(z) + \sum_{n=1}^{\infty} \big(L_n f(z) - L_{n-1} f(z)\big) = \lim_{n\to\infty} L_n f(z) = f(z)\end{aligned}$$

and

$$D^\alpha Lf(z) = D^\alpha L_0 f(z) + \sum_{n=1}^{\infty} \big(D^\alpha L_n f(z) - D^\alpha L_{n-1} f(z)\big) = 0,$$

because $u_n \equiv 1$ in a neighbourhood of E, $f_{|E} \in s(E)$ and the polynomials $L_n f$ are holomorphic.

We conclude that $L : s(E) \to \mathcal{A}^\infty(E)$ is a continuous and linear extension operator as required.

($\Longleftarrow$) By the assumption there exists a continuous and linear extension operator $L : s(E) \to \mathcal{A}^\infty(E)$. Continuity means that

$$\begin{gathered}\forall K \subset\subset \mathbb{C} \quad \forall \alpha \in \mathbb{Z}_+^2 \quad \exists k \in \mathbb{N} \quad \exists a_{-1}, a_0, \ldots, a_k \geq 0 \quad \forall f \in s(E) \quad : \\ \|D^\alpha Lf\|_K \leq a_{-1} \cdot |f|_{-1} + a_0 \cdot |f|_0 + \ldots + a_k \cdot |f|_k.\end{gathered}$$

Since $|f|_{-1} \leq |f|_0 \leq |f|_1 \leq |f|_2 \leq \ldots$, we have

$$\forall K \subset\subset \mathbb{C} \quad \forall \alpha \in \mathbb{Z}_+^2 \quad \exists k \in \mathbb{N} \quad \exists M \geq 0 \quad \forall f \in s(E) \quad : \quad \|D^\alpha Lf\|_K \leq M \cdot |f|_k$$

and specifically, by considering only polynomials and taking $K := E$ and $\alpha := (1, 0)$, we obtain

$$\exists k \in \mathbb{N} \quad \exists M \geq 0 \quad \forall n \in \mathbb{N} \quad \forall p \in \mathcal{P}_n \quad : \quad \left\| \frac{\partial L(p_{|E})}{\partial z} \right\|_E \leq M \cdot |p_{|E}|_k.$$

Fix arbitrary $n \in \mathbb{N}$ and $p \in \mathcal{P}_n$. Because $\operatorname{dist}_E(p, \mathcal{P}_j) = 0$ for $j \geq n$, we see that

$$|p_{|E}|_k = \|p\|_E + \sup_{j \in \mathbb{N}} j^k \cdot \operatorname{dist}_E(p, \mathcal{P}_j) = \|p\|_E + \max_{1 \leq j \leq n-1} j^k \cdot \operatorname{dist}_E(p, \mathcal{P}_j) \leq$$
$$\leq \|p\|_E + (n-1)^k \cdot \max_{1 \leq j \leq n-1} \operatorname{dist}_E(p, \mathcal{P}_j) \leq \|p\|_E + (n-1)^k \cdot \|p\|_E \leq n^k \cdot \|p\|_E.$$

Furthermore, since the set E is assumed to be $\mathcal{A}^\infty$-determining and we have there $L(p_{|E})_{|E} \equiv p_{|E}$, then also

$$\left(\frac{\partial L(p_{|E})}{\partial z} \right)_{|E} \equiv (p')_{|E}. \tag{1}$$

Combining these three estimates we obtain

$$\|p'\|_E = \left\| \frac{\partial L(p_{|E})}{\partial z} \right\|_E \leq M \cdot |p_{|E}|_k \leq M \cdot n^k \cdot \|p\|_E,$$

which proves GMI. $\square$

Remark 9.3. A careful inspection of the constants in the proof of the previous theorem reveals that for all $\alpha \in \mathbb{Z}_+^2 \setminus \{(0,0)\}$ we have

$$\widetilde{C}_\alpha \leq \sum_{\beta_1 \leq \alpha_1} \binom{\alpha_1}{\beta_1} \cdot \big(d \cdot (\beta_1 + \alpha_2)\big)^{4 \cdot (\beta_1 + \alpha_2)} \cdot \widetilde{M} \cdot M^{\alpha_1 - \beta_1} \leq$$
$$\leq \sum_{\beta_1 \leq \alpha_1} \binom{\alpha_1}{\beta_1} \cdot \big(d \cdot |\alpha|\big)^{4 \cdot |\alpha|} \cdot \widetilde{M} \cdot M^{|\alpha|} \leq \Big(2d \cdot \widetilde{M} \cdot M \cdot |\alpha|\Big)^{4 \cdot |\alpha|},$$

$$C_{|\alpha|} + 5\widetilde{C}_\alpha + \widetilde{C}_\alpha \cdot S_{|\alpha|} \leq \big(d \cdot |\alpha|\big)^{4 \cdot |\alpha|} + \Big(2d \cdot \widetilde{M} \cdot M \cdot |\alpha|\Big)^{4 \cdot |\alpha|} \cdot \left(5 + \frac{7\pi^2}{6} \cdot 2^{k \cdot |\alpha|}\right) \leq$$
$$\leq \Big(2d \cdot \widetilde{M} \cdot M \cdot |\alpha|\Big)^{4 \cdot |\alpha|} \cdot \left(6 + \frac{7\pi^2}{6} \cdot 2^{k \cdot |\alpha|}\right) \leq \Big(2d \cdot \widetilde{M} \cdot M \cdot |\alpha|\Big)^{4 \cdot |\alpha|} \cdot \big(6 + 12 \cdot 2^k\big)^{|\alpha|} \leq \big(d_1 \cdot |\alpha|\big)^{4 \cdot |\alpha|},$$

while

$$C_0 + 5\widetilde{C}_{(0,0)} + \widetilde{C}_{(0,0)} \cdot S_0 \leq d + 5d \cdot \widetilde{M} + d \cdot \widetilde{M} \cdot \frac{7\pi^2}{6} \leq 18d \cdot \widetilde{M} \leq d_1,$$

where $d \geq 1$ is the absolute constant from proposition 6.8 on cutoff functions and $d_1 := 9d \cdot \widetilde{M} \cdot M \cdot 2^k$ depends only on the set E. Consequently for all $f \in s(E)$ and $\alpha \in \mathbb{Z}_+^2 \setminus \{(0,0)\}$ we have

$$\|D^\alpha L f\|_{\mathbb{C}} \leq (d_1 \cdot |\alpha|)^{4 \cdot |\alpha|} \cdot |f|_{k \cdot |\alpha| + 3},$$

$$\|Lf\|_{\mathbb{C}} \leq d_1 \cdot |f|_3.$$

Remark 9.4. Note that the operator L constructed in theorem 9.2 preserves each polynomial in a certain open neighbourhood of the set E. Indeed, if $p \in \mathcal{P}_n$ for some $n \in \mathbb{N}$, then for all $z \in U_n := \left\{z \in \mathbb{C} : \operatorname{dist}(z, E) < \frac{1}{8n^k}\right\} \subset \bigcap_{j=1}^n \{z \in \mathbb{C} : u_j(z) = 1\}$ we see that

$$L(p_{|E})(z) = u_1(z) \cdot L_0(p_{|E})(z) + \sum_{j=1}^{\infty} u_j(z) \cdot \big(L_j(p_{|E})(z) - L_{j-1}(p_{|E})(z)\big) =$$
$$= u_1(z) \cdot L_0(p_{|E})(z) + \sum_{j=1}^{n} u_j(z) \cdot \big(L_j(p_{|E})(z) - L_{j-1}(p_{|E})(z)\big) =$$
$$= L_0(p_{|E})(z) + \sum_{j=1}^{n} \big(L_j(p_{|E})(z) - L_{j-1}(p_{|E})(z)\big) = L_n(p_{|E})(z) = p(z),$$

because for all $j \geq n$ we have $L_j(p_{|E}) \equiv p$.

Note also that in the proof of the first implication ($\Longrightarrow$), we needed the assumption that the set E is polynomially convex only in order to deduce that it is $\mathcal{A}^\infty$-determining. In fact for any set admitting GMI, it is possible to construct a continuous and linear extension operator as in the theorem.

In the converse proof ($\Longleftarrow$), we needed the assumption that the set E is $\mathcal{A}^\infty$-determining only in order to obtain equality (1). Instead we could also assume that the operator L preserves each polynomial in a certain open neighbourhood of the set E. Without either one of these two assumptions, the implication ($\Longleftarrow$) would not be true. To see this consider the set $E := \{0\}$, for which $s(E) \cong \mathbb{C}$, and the operator

$$L : s(E) \ni f \longrightarrow Lf \in \mathcal{A}^\infty(E) \qquad Lf \equiv f(0).$$

Then we have

$$\forall K \subset\subset \mathbb{C} \quad \forall \alpha \in \mathbb{Z}_+^2 \quad \forall f \in s(E) \quad : \quad \|D^\alpha Lf\|_K \leq |f(0)| = |f|_{-1},$$

which demonstrates the continuity of the operator L.

Corollary 9.5. *For any polynomially convex compact set $E \subset\subset \mathbb{C}$ we have*

$$\text{(i)} \qquad \mathcal{H}^\infty(E)_{|E} \subset s(E) \cap \mathcal{A}^\infty(E)_{|E} \subset \mathcal{C}^\infty(E),$$

while if it additionally admits GMI*, then we have*

$$\text{(ii)} \qquad \mathcal{H}^\infty(E)_{|E} \subset s(E) \subset \mathcal{A}^\infty(E)_{|E} \subset \mathcal{C}^\infty(E).$$

Furthermore for any compact set $E \subset\subset \mathbb{R}$ we have

$$\text{(iii)} \qquad \mathcal{H}^\infty(E)_{|E} \subset \mathcal{A}^\infty(E)_{|E} = \mathcal{C}^\infty(E) \subset s(E),$$

while if it additionally admits GMI*, then we have*

$$\text{(iv)} \qquad \mathcal{H}^\infty(E)_{|E} \subset \mathcal{A}^\infty(E)_{|E} = \mathcal{C}^\infty(E) = s(E).$$

Proof. The inclusions $\mathcal{H}^\infty(E)_{|E} \subset \mathcal{A}^\infty(E)_{|E} \subset \mathcal{C}^\infty(E)$ are trivially true for any set $E \subset\subset \mathbb{C}$, while the inclusion $\mathcal{H}^\infty(E)_{|E} \subset s(E)$ follows from corollary 8.20, provided that the set E is polynomially convex. Additionally the inclusion $s(E) \subset \mathcal{A}^\infty(E)_{|E}$, which is essentially S.N. Bernstein's theorem [cf. Pleśniak 1, theorem 3.3.iii; cf. Bernstein 1], follows from theorem 9.2 for any set $E \subset\subset \mathbb{C}$ admitting GMI (even if it is not polynomially convex - see remark 9.4) and this finishes statements (i) and (ii).

Furthermore from corollary 8.6 following Jackson's theorem we know that $\mathcal{C}^\infty(E) \subset s(E)$ for any compact set $E \subset\subset \mathbb{R}$ and together with statement (ii), this leads to statement (iv). Finally, if $E \subset\subset \mathbb{R}$, then the interval $I := \operatorname{conv} E$ admits GMI and we see that

$$\mathcal{C}^\infty(E) = \mathcal{C}^\infty(I)_{|E} = \left(\mathcal{A}^\infty(I)_{|I}\right)_{|E} = \mathcal{A}^\infty(I)_{|E} \subset \mathcal{A}^\infty(E)_{|E},$$

which completes statement (iii). Note that this last inclusion can also be proved by solving a simple differential equation. $\square$

Example 9.6. Without the assumption that the compact set $E \subset\subset \mathbb{R}$ admits GMI, the inclusion $s(E) \subset \mathcal{C}^\infty(E)$ does not have to be true. Consider for example the set $E = \{0\} \cup \bigcup_{j=1}^\infty \left\{-\frac{1}{2^j}, \frac{1}{2^j}\right\}$ and the function $f(x) := |x|$. Clearly $f \notin \mathcal{C}^\infty(E)$ but we will show that $f \in s(E)$.

Indeed consider the following Lagrange interpolation polynomials for $n \in \mathbb{N}$

$$p_n(x) := \sum_{\mu=1,\ldots,n} 2^\mu \cdot x^2 \cdot \prod_{\substack{\nu=1,\ldots,n \\ \nu \neq \mu}} \frac{4^\nu \cdot x^2 - 1}{4^{\nu-\mu} - 1}.$$

Note that $p_n \in \mathcal{P}_{2n}$ and $p_n(x) = f(x)$ whenever $x = 0$ or $|x| = \frac{1}{2^j}$ for some $j \in \{1, \ldots, n\}$. Therefore we have

$$\operatorname{dist}_E(f, \mathcal{P}_{2n}) \leq \|f - p_n\|_E = \|f - p_n\|_{E \cap B\left(0, \frac{1}{2^{n+1}}\right)} \leq \|f\|_{E \cap B\left(0, \frac{1}{2^{n+1}}\right)} + \|p_n\|_{E \cap B\left(0, \frac{1}{2^{n+1}}\right)}.$$

For any $x \in E \cap B\left(0, \frac{1}{2^{n+1}}\right)$ we have $|f(x)| = |x| \leq \frac{1}{2^{n+1}}$ and

$$|p_n(x)| \leq \sum_{\mu=1,\ldots,n} 2^\mu \cdot \frac{1}{4^{n+1}} \cdot \prod_{\substack{\nu=1,\ldots,n\\ \nu\neq\mu}} \left|\frac{4^\nu \cdot x^2 - 1}{4^{\nu-\mu} - 1}\right| \leq \sum_{\mu=1,\ldots,n} 2^\mu \cdot \frac{1}{4^{n+1}} \cdot \prod_{\substack{\nu=1,\ldots,n\\ \nu\neq\mu}} \left|\frac{1}{\frac{1}{4} - 1}\right| =$$

$$= \sum_{\mu=1,\ldots,n} 2^\mu \cdot \frac{1}{4^{n+1}} \cdot \left(\frac{4}{3}\right)^{n-1} \leq 2^{n+1} \cdot \frac{1}{4^{n+1}} \cdot \left(\frac{4}{3}\right)^{n-1} = \frac{3}{8} \cdot \left(\frac{2}{3}\right)^n.$$

We conclude that

$$\mathrm{dist}_E(f, \mathcal{P}_{2n}) \leq \frac{1}{2^{n+1}} + \frac{3}{8} \cdot \left(\frac{2}{3}\right)^n \leq \frac{7}{8} \cdot \left(\frac{2}{3}\right)^n$$

from which it follows that $f \in s(E)$. $\square$

Remark 9.7 [cf. Pleśniak 1, theorem 3.3.iv-vi]. We saw already that if a compact set $E \subset\subset \mathbb{R}$ admits GMI, then $\mathcal{A}^\infty(E)_{|E} = \mathcal{C}^\infty(E) = s(E)$. This allowed W. Pleśniak to assert in his extension theorem three equivalent topological properties of the function spaces $s(E)$ and $\mathcal{C}^\infty(E)$ with their respective Jackson and quotient topologies. However without additional assumptions there is no obvious analogy in the complex case.

Remark 9.8. The characterization of compact sets $E \subset\subset \mathbb{C}$, for which $\mathcal{A}^\infty(E)_{|E} = s(E)$, remains an open problem, especially for totally disconnected sets. In [Siciak 3, theorem 1.10], J. Siciak proved this property for simply connected continua for which the conformal mapping $\psi : \hat{\mathbb{C}} \setminus B(0,1) \longrightarrow \hat{\mathbb{C}} \setminus E$, with $\psi(\infty) = \infty$, is Hölder-continuous in the annulus $\{z \in \mathbb{C} : 1 \leq |z| \leq 2\}$. Subsequently we were able to generalize this result for a finite union of such disjoint simply connected continua. More recently L. Gendre constructed an approximation technique for functions of the class $\mathcal{A}^\infty(E)_{|E}$, where $E \subset\subset \mathbb{C}^N$, $N \in \mathbb{N}$, is Whitney 1-regular and admits HCP as well as ŁS, which allowed him to assert that $\mathcal{A}^\infty(E)_{|E} = s(E)$ [Gendre, corollary 7]. It remains to be verified whether the assumption of Whitney 1-regularity can be somehow circumvented in the case of sets on the complex plane.

CHAPTER X

EXTENSION PROPERTY BY BOS-MILMAN (EXP)

L.P. Bos and P.D. Milman, adapting earlier work by W. Pleśniak, formulated a different extension theorem for compact subsets of $\mathbb{R}^N$, $N \in \mathbb{N}$. They proved that any such set admitting GMI also admits a "bounded extension of $\mathcal{C}^\infty$ functions with homogeneous linear loss of differentiability (in the quotient topology)". We modified their definition of the extension property so that it can be deduced from GMI for any polynomially convex compact subset of $\mathbb{C}$. In this definition we replaced the quotient norms $|\!|\!| \cdot |\!|\!|_{E,m\cdot\ell}$, where $m \in \mathbb{N}$, with Jackson norms $|\cdot|_{k\cdot\ell+c_0}$, where $k \geq 1$, as they work well for functions that are holomorphic in some open neighbourhood of a polynomially convex compact set.

Definition 10.1 [cf. Eggink, definition 9.1; cf. Bos-Milman, definition 3.10]. A compact set $E \subset\subset \mathbb{C}$ admits the Extension Property by Bos-Milman EXP(k,u), where $k,u \geq 1$, if it is $\mathcal{A}^\infty$-determining and

$$\begin{aligned} &\forall f \in s(E) \quad \exists \widetilde{f} \in \mathcal{A}^\infty(E) \quad : \\ &(a) \quad \widetilde{f}_{|E} \equiv f, \\ &(b) \quad \|\widetilde{f}\|_{\mathbb{C},0} \leq c_2 \cdot ||f||_E, \\ &(c) \quad \|\widetilde{f}\|_{\mathbb{C},\ell} \leq (c_2 \cdot \ell^u)^{\ell+c_1} \cdot |f|_{k\cdot\ell+c_0} \quad \text{for all } \ell \in \mathbb{N}, \end{aligned}$$

with some $c_0, c_1 \geq 0$ and $c_2 \geq 1$ dependent only on the set E. We will write that the set E admits EXP, if it admits EXP(k,u), for some $k,u \geq 1$.

Remark 10.2. In remark 9.3 following Pleśniak's theorem 9.2 we showed similar properties of the extension Lf. However here we want to estimate the norm $\|\cdot\|_{\mathbb{C}}$ of the extension by the usual norm $\|\cdot\|_E$ of the function itself, rather than $|\cdot|_3$ or even $\|\cdot\|_{E_\delta}$ for some $\delta > 0$. In order to achieve this L.P. Bos and P.D. Milman modified W. Pleśniak's proof, however at the expense of the linearity of the extension operator.

By the way, L.P. Bos and P.D. Milman have also proved the existence of a "bounded **linear** extension of $\mathcal{C}^\infty$ functions with homogeneous linear loss of differentiability (in the quotient topology)" for compact subsets of $\mathbb{R}^N$, $N \in \mathbb{N}$, admitting "quasi-geometric local bounds on polynomials", which in turn is equivalent to LMP. It appears that a corresponding theorem is true for compact subsets of the complex plane too.

Proposition 10.3 [Bos-Milman, proof of theorem B]. *For any compact set $E \subset\subset \mathbb{C}$ and coefficient $k \geq 1$ there exists a sequence of decreasing cutoff functions $\widetilde{u}_n \in \mathcal{C}^\infty(\mathbb{C})$ such that for all $n \in \mathbb{N}$*

$$\begin{aligned} &(i) \quad 0 \leq \widetilde{u}_{n+1}(z) \leq \widetilde{u}_n(z) \leq 1 && \textit{for all } z \in \mathbb{C}, \\ &(ii) \quad \widetilde{u}_n(z) = 1 && \textit{if } \operatorname{dist}(z,E) \leq \frac{1}{8n^k}, \\ &(iii) \quad \widetilde{u}_n(z) = 0 && \textit{if } \operatorname{dist}(z,E) \geq \frac{1}{n^k}, \\ &(iv) \quad \|D^\alpha \widetilde{u}_n\|_{\mathbb{C}} \leq C_{|\alpha|} \cdot n^{(k+1)\cdot|\alpha|} && \textit{for all } \alpha = (\alpha_1,\alpha_2) \in \mathbb{Z}_+^2 \end{aligned}$$

where $C_t := (d \cdot t)^{4t}$ for $t \in \mathbb{N}$, $C_0 := d$ and $d \geq 1$ is the absolute constant from proposition 6.8.

Proof. For each $j \in \mathbb{N}$ denote by $u_j \in \mathcal{C}^\infty(\mathbb{C})$ the cutoff function constructed in proposition 6.8 for the compact set $K := E$ and radius $\epsilon_j := \frac{1}{j^k}$, i.e.

$$\begin{array}{lll} (a) & 0 \le u_j(z) \le 1 & \text{for all } z \in \mathbb{C}, \\ (b) & u_j(z) = 1 & \text{if } \operatorname{dist}(z,E) \le \dfrac{\epsilon_j}{8}, \\ (c) & u_j(z) = 0 & \text{if } \operatorname{dist}(z,E) \ge \epsilon_j, \\ (d) & \|D^\alpha u_j\|_{\mathbb{C}} \le \dfrac{C_{|\alpha|}}{\epsilon_j^{|\alpha|}} & \text{for all } \alpha = (\alpha_1,\alpha_2) \in \mathbb{Z}_+^2 \end{array}$$

where $C_t := (d \cdot t)^{4t}$ for $t \in \mathbb{N}$, $C_0 := d$ and $d \ge 1$ is some absolute constant. We define $\widetilde{u}_n \in \mathcal{C}^\infty(\mathbb{C})$ as follows:

$$\widetilde{u}_n(z) := \prod_{j=1,\dots,n} u_j(z)$$

and we see that conditions (i), (ii) and (iii) are obviously fulfilled. In order to verify condition (iv), we note that by the Leibniz rule we have for $\alpha \in \mathbb{Z}_+^2$

$$D^\alpha \widetilde{u}_n = \sum_{\gamma_1+\cdots+\gamma_n=\alpha} a_\gamma \cdot D^{\gamma_1} u_1 \cdot \ldots \cdot D^{\gamma_n} u_n$$

with some combinatorial constants $a_\gamma \in \mathbb{N}$, where $\gamma = (\gamma_1,\dots,\gamma_n)$ and $\gamma_1,\dots,\gamma_n \in \mathbb{Z}_+^2$. We conclude that

$$\begin{aligned} \|D^\alpha \widetilde{u}_n\|_{\mathbb{C}} &\le \sum_{\gamma_1+\ldots+\gamma_n=\alpha} a_\gamma \cdot \prod_{j=1,\dots,n} \|D^{\gamma_j} u_j\|_{\mathbb{C}} \le \sum_{\gamma_1+\ldots+\gamma_n=\alpha} a_\gamma \cdot \prod_{j=1,\dots,n} \frac{C_{|\gamma_j|}}{\epsilon_j^{|\gamma_j|}} \le \\ &\le \sum_{\gamma_1+\ldots+\gamma_n=\alpha} a_\gamma \cdot \prod_{j=1,\dots,n} \frac{(d\cdot|\gamma_j|)^{4\cdot|\gamma_j|}}{\epsilon_j^{|\gamma_j|}} \le \sum_{\gamma_1+\ldots+\gamma_n=\alpha} a_\gamma \cdot \prod_{j=1,\dots,n} \frac{(d\cdot|\alpha|)^{4\cdot|\gamma_j|}}{\epsilon_n^{|\gamma_j|}} = \\ &= \sum_{\gamma_1+\ldots+\gamma_n=\alpha} a_\gamma \cdot \frac{(d\cdot|\alpha|)^{4\cdot|\alpha|}}{\epsilon_n^{|\alpha|}} \le n^{|\alpha|} \cdot \frac{(d\cdot|\alpha|)^{4\cdot|\alpha|}}{\epsilon_n^{|\alpha|}} = C_{|\alpha|} \cdot n^{(k+1)\cdot|\alpha|}, \end{aligned}$$

because $\sum_{\gamma_1+\ldots+\gamma_n=\alpha} a_\gamma \le n^{|\alpha|}$, which follows from the proof of the Leibniz rule by induction. $\square$

THEOREM 10.4 [cf. Eggink, theorem 9.2; cf. Bos-Milman, theorem B]. *For any polynomially convex compact set $E \subset\subset \mathbb{C}$, $k' > k \ge 1$ and $u \ge 1$ we have*

$$\begin{aligned} \mathrm{GMI}(k) &\implies \mathrm{EXP}(k+1,4), \\ \mathrm{EXP}(k,u) &\implies \mathrm{GMI}(k'). \end{aligned}$$

PROOF. Let's first assume that the set E admits GMI(k), i.e.

$$\exists M \ge 1 \quad \forall n \in \mathbb{N} \quad \forall p \in \mathcal{P}_n \quad : \quad \|p'\|_E \le M \cdot n^k \cdot \|p\|_E.$$

Proposition 1.18 implies that

$$\forall n \in \mathbb{N} \quad \forall p \in \mathcal{P}_n \quad : \quad \|p\|_{E_{1/n^k}} \le \widetilde{M} \cdot \|p\|_E,$$

where $\widetilde{M} := e^M$. From propositions 1.21 and 5.4 we know that the set E is $\mathcal{A}^\infty$-determining.

Now we fix a function $f \in s(E)$ and for each $n \in \mathbb{Z}_+$ we take $p_n \in \mathcal{P}_n$ to be any polynomial of best approximation, i.e. $\|f - p_n\|_E = \operatorname{dist}_E(f, \mathcal{P}_n)$. We define the desired extension $\widetilde{f}$ as follows:

$$\widetilde{f} := \widetilde{u}_1 \cdot p_0 + \sum_{n=1}^\infty \widetilde{u}_n \cdot (p_n - p_{n-1}),$$

where $\widetilde{u}_n \in \mathcal{C}^\infty(\mathbb{C})$, $n \in \mathbb{N}$, is a sequence of cutoff functions as constructed in proposition 10.3 for the compact set E and coefficient k.

Similarly to the method used by W. Pleśniak, L.P. Bos and P.D. Milman show that this series is convergent together with all its derivatives. For this purpose let's fix $\alpha \in \mathbb{Z}_+^2$ and $n \in \mathbb{N}$. We use the Leibniz rule to see that

$$\|D^\alpha\big(\widetilde{u}_n \cdot (p_n - p_{n-1})\big)\|_{\mathbb{C}} \le \sum_{\substack{\beta\in\mathbb{Z}_+^2\\ \beta\le\alpha}} \binom{\alpha}{\beta} \cdot \|D^\beta \widetilde{u}_n \cdot D^{\alpha-\beta}(p_n - p_{n-1})\|_{\mathbb{C}} =$$
$$= \sum_{\beta\le\alpha} \binom{\alpha}{\beta} \cdot \|D^\beta \widetilde{u}_n \cdot D^{\alpha-\beta}(p_n - p_{n-1})\|_{E_{1/n^k}} \le \dots$$

since $u_n \equiv 0$ outside of E_{1/n^k},

$$\dots \le \sum_{\beta\le\alpha} \binom{\alpha}{\beta} \cdot \|D^\beta \widetilde{u}_n\|_{E_{1/n^k}} \cdot \|D^{\alpha-\beta}(p_n - p_{n-1})\|_{E_{1/n^k}} \le$$
$$\le \sum_{\beta\le\alpha} \binom{\alpha}{\beta} \cdot C_{|\beta|} \cdot n^{(k+1)\cdot|\beta|} \cdot \widetilde{M} \cdot \|D^{\alpha-\beta}(p_n - p_{n-1})\|_E \le \dots$$

because of the properties of the cutoff functions $\widetilde{u}_n$ and the fact that $D^{\alpha-\beta}(p_n - p_{n-1})$ is a holomorphic polynomial of degree n at most,

$$\dots \le \sum_{\substack{\beta\le\alpha\\ \beta_2=\alpha_2}} \binom{\alpha}{\beta} \cdot C_{|\beta|} \cdot n^{(k+1)\cdot|\beta|} \cdot \widetilde{M} \cdot M^{\alpha_1-\beta_1} \cdot n^{k\cdot(\alpha_1-\beta_1)} \cdot \|p_n - p_{n-1}\|_E \le$$
$$\le \sum_{\substack{\beta\le\alpha\\ \beta_2=\alpha_2}} \binom{\alpha}{\beta} \cdot C_{|\beta|} \cdot \widetilde{M} \cdot M^{\alpha_1-\beta_1} \cdot n^{(k+1)\cdot|\alpha|} \cdot \|p_n - p_{n-1}\|_E = \widetilde{C}_\alpha \cdot n^{(k+1)\cdot|\alpha|} \cdot \|p_n - p_{n-1}\|_E,$$

where $\widetilde{C}_\alpha := \sum_{\beta_1\le\alpha_1} \binom{\alpha_1}{\beta_1} \cdot C_{\beta_1+\alpha_2} \cdot \widetilde{M} \cdot M^{\alpha_1-\beta_1}$ are constants depending solely on the set E.

Furthermore we have

$$\|p_n - p_{n-1}\|_E = \|(p_n - f) - (p_{n-1} - f)\|_E \le \|f - p_n\|_E + \|f - p_{n-1}\|_E =$$
$$= \operatorname{dist}_E(f, \mathcal{P}_n) + \operatorname{dist}_E(f, \mathcal{P}_{n-1}) \le 2 \cdot \operatorname{dist}_E(f, \mathcal{P}_{n-1}),$$

and therefore

$$\|D^\alpha\big(\widetilde{u}_n \cdot (p_n - p_{n-1})\big)\|_{\mathbb{C}} \le 2\widetilde{C}_\alpha \cdot n^{(k+1)\cdot|\alpha|} \cdot \operatorname{dist}_E(f, \mathcal{P}_{n-1}),$$

which for all $n \ge 2$ leads us to

$$\|D^\alpha\big(\widetilde{u}_n \cdot (p_n - p_{n-1})\big)\|_{\mathbb{C}} \le 2\widetilde{C}_\alpha \cdot \frac{n^{(k+1)\cdot|\alpha|}}{(n-1)^{(k+1)\cdot|\alpha|+2}} \cdot |f|_{(k+1)\cdot|\alpha|+2}.$$

Finally we obtain

$$\|D^\alpha \widetilde{f}\|_{\mathbb{C}} \le \|D^\alpha\big(\widetilde{u}_1 \cdot p_0\big)\|_{\mathbb{C}} + \sum_{n=1}^{\infty} \|D^\alpha\big(\widetilde{u}_n \cdot (p_n - p_{n-1})\big)\|_{\mathbb{C}} \le$$
$$\le C_{|\alpha|} \cdot \|f\|_E + 2\widetilde{C}_\alpha \cdot \operatorname{dist}_E(f, \mathcal{P}_0) + \widetilde{C}_\alpha \cdot \widetilde{S}_{|\alpha|} \cdot |f|_{(k+1)\cdot|\alpha|+2} \le$$
$$\le \Big(C_{|\alpha|} + 2\widetilde{C}_\alpha + \widetilde{C}_\alpha \cdot \widetilde{S}_{|\alpha|}\Big) \cdot |f|_{(k+1)\cdot|\alpha|+2},$$

where for $t \in \mathbb{Z}_+$ we put

$$\widetilde{S}_t := \sum_{n=2}^{\infty} \frac{2n^{(k+1)\cdot t}}{(n-1)^{(k+1)\cdot t+2}} \le \sum_{n=2}^{\infty} \frac{2 \cdot 2^{(k+1)\cdot t} \cdot (n-1)^{(k+1)\cdot t}}{(n-1)^{(k+1)\cdot t+2}} =$$
$$= 2^{1+(k+1)\cdot t} \cdot \sum_{n=2}^{\infty} \frac{1}{(n-1)^2} = \frac{\pi^2}{3} \cdot 2^{(k+1)\cdot t} < +\infty.$$

A careful inspection of the constants reveals that for all $\alpha \in \mathbb{Z}_+^2 \setminus \{(0,0)\}$ we have

$$\widetilde{C}_\alpha \le \sum_{\beta_1 \le \alpha_1} \binom{\alpha_1}{\beta_1} \cdot \big(d \cdot (\beta_1 + \alpha_2)\big)^{4\cdot(\beta_1+\alpha_2)} \cdot \widetilde{M} \cdot M^{\alpha_1 - \beta_1} \le$$
$$\le \sum_{\beta_1 \le \alpha_1} \binom{\alpha_1}{\beta_1} \cdot \big(d \cdot |\alpha|\big)^{4\cdot|\alpha|} \cdot \widetilde{M} \cdot M^{|\alpha|} \le \Big(2d \cdot \widetilde{M} \cdot M \cdot |\alpha|\Big)^{4\cdot|\alpha|},$$

$$C_{|\alpha|} + 2\widetilde{C}_\alpha + \widetilde{C}_\alpha \cdot \widetilde{S}_{|\alpha|} \le \big(d \cdot |\alpha|\big)^{4\cdot|\alpha|} + \Big(2d \cdot \widetilde{M} \cdot M \cdot |\alpha|\Big)^{4\cdot|\alpha|} \cdot \left(2 + \frac{\pi^2}{3} \cdot 2^{(k+1)\cdot|\alpha|}\right) \le$$
$$\le \Big(2d \cdot \widetilde{M} \cdot M \cdot |\alpha|\Big)^{4\cdot|\alpha|} \cdot \left(3 + \frac{\pi^2}{3} \cdot 2^{(k+1)\cdot|\alpha|}\right) \le \Big(2d \cdot \widetilde{M} \cdot M \cdot |\alpha|\Big)^{4\cdot|\alpha|} \cdot \big(3 + 7 \cdot 2^k\big)^{|\alpha|} \le \big(d_1 \cdot |\alpha|\big)^{4\cdot|\alpha|},$$

while

$$C_0 + 2\widetilde{C}_{(0,0)} + \widetilde{C}_{(0,0)} \cdot \widetilde{S}_0 \le d + 2d \cdot \widetilde{M} + d \cdot \widetilde{M} \cdot \frac{\pi^2}{3} \le 7d \cdot \widetilde{M} \le d_1,$$

where $d \ge 1$ is the absolute constant from proposition 6.8 on cutoff functions and $d_1 := 4d \cdot \widetilde{M} \cdot M \cdot 2^k$ depends only on the set E. Consequently for all $\ell \in \mathbb{N}$ we obtain

$$\|\widetilde{f}\|_{\mathbb{C},\ell} = \|\widetilde{f}\|_{\mathbb{C}} + \sum_{|\alpha|=\ell} \|D^\alpha \widetilde{f}\|_{\mathbb{C}} \le$$
$$\le \Big(C_0 + 2\widetilde{C}_{(0,0)} + \widetilde{C}_{(0,0)} \cdot \widetilde{S}_0\Big) \cdot |f|_2 + \sum_{|\alpha|=\ell} \Big(C_{|\alpha|} + 2\widetilde{C}_\alpha + \widetilde{C}_\alpha \cdot \widetilde{S}_{|\alpha|}\Big) \cdot |f|_{(k+1)\cdot|\alpha|+2} \le$$
$$\le d_1 \cdot |f|_2 + \sum_{|\alpha|=\ell} \big(d_1 \cdot |\alpha|\big)^{4\cdot|\alpha|} \cdot |f|_{(k+1)\cdot|\alpha|+2} \le \Big(d_1 + (\ell+1) \cdot \big(d_1 \cdot \ell\big)^{4\ell}\Big) \cdot |f|_{(k+1)\cdot\ell+2} \le$$
$$\le (\ell+2) \cdot \big(d_1 \cdot \ell\big)^{4\ell} \cdot |f|_{(k+1)\cdot\ell+2} \le \big(c_2 \cdot \ell^4\big)^\ell \cdot |f|_{(k+1)\cdot\ell+c_0} < +\infty,$$

where $c_0 := 2$ and $c_2 := \max\Big\{3d_1^4,\, 2\widetilde{M}\Big\}$ depend only on the set E.

Now we know that $\widetilde{f} \in \mathcal{C}^\infty(\mathbb{C})$ we will show that $\widetilde{f}_{|E} \equiv f$ and $\widetilde{f} \in \mathcal{A}^\infty(E)$. For this purpose let's fix $z \in E$ and $\alpha \in \mathbb{Z}_+^2$ such that $\alpha_2 \ge 1$. It is easily seen that

$$\widetilde{f}(z) = \widetilde{u}_1(z) \cdot p_0(z) + \sum_{n=1}^{\infty} \widetilde{u}_n(z) \cdot \big(p_n(z) - p_{n-1}(z)\big) =$$
$$= p_0(z) + \sum_{n=1}^{\infty} \big(p_n(z) - p_{n-1}(z)\big) = \lim_{n\to\infty} p_n(z) = f(z)$$

and

$$D^\alpha \widetilde{f}(z) = D^\alpha p_0(z) + \sum_{n=1}^{\infty} \big(D^\alpha p_n(z) - D^\alpha p_{n-1}(z)\big) = 0,$$

because $\widetilde{u}_n \equiv 1$ in an open neighbourhood of the set E, $f \in s(E)$ and the polynomials p_n are holomorphic.

Finally for $\alpha = (0,0)$ we will show something better. For $\nu \in \mathbb{N}$ denote by $S_\nu \in \mathcal{C}^\infty(\mathbb{C})$ the partial sum of the series $\widetilde{f}$:

$$S_\nu := \widetilde{u}_1 \cdot p_0 + \sum_{n=1}^{\nu} \widetilde{u}_n \cdot (p_n - p_{n-1}) = \widetilde{u}_1 \cdot p_0 + \sum_{n=1}^{\nu} \widetilde{u}_n \cdot p_n - \sum_{n=1}^{\nu} \widetilde{u}_n \cdot p_{n-1} =$$
$$= \widetilde{u}_1 \cdot p_0 + \sum_{n=1}^{\nu} \widetilde{u}_n \cdot p_n - \sum_{n=0}^{\nu-1} \widetilde{u}_{n+1} \cdot p_n = \sum_{n=1}^{\nu} \widetilde{u}_n \cdot p_n - \sum_{n=1}^{\nu-1} \widetilde{u}_{n+1} \cdot p_n = \widetilde{u}_\nu \cdot p_\nu + \sum_{n=1}^{\nu-1} p_n \cdot (\widetilde{u}_n - \widetilde{u}_{n+1}).$$

Since $\operatorname{supp}\widetilde{u}_{n+1} \subset \operatorname{supp}\widetilde{u}_n \subset E_{1/n^k}$ for all $n \in \mathbb{N}$ we see that

$$\|p_n\|_{E_{1/n^k}} \le \widetilde{M} \cdot \|p_n\|_E \le \widetilde{M} \cdot \big(\|f - p_n\|_E + \|f\|_E\big) = \widetilde{M} \cdot \big(\operatorname{dist}_E(f, \mathcal{P}_n) + \|f\|_E\big) \le 2\widetilde{M} \cdot \|f\|_E$$

and for any $z \in \mathbb{C}$ we have

$$|\widetilde{u}_\nu(z) \cdot p_\nu(z)| \le |\widetilde{u}_\nu(z)| \cdot \|p_\nu\|_{E_{1/\nu^k}} \le 2\widetilde{M} \cdot \|f\|_E \cdot \widetilde{u}_\nu(z),$$

$$|p_n(z) \cdot \big(\widetilde{u}_n(z) - \widetilde{u}_{n+1}(z)\big)| \le \|p_n\|_{E_{1/n^k}} \cdot |\widetilde{u}_n(z) - \widetilde{u}_{n+1}(z)| \le 2\widetilde{M} \cdot \|f\|_E \cdot \big(\widetilde{u}_n(z) - \widetilde{u}_{n+1}(z)\big),$$

by the assumption that the sequence $\widetilde{u}_n$ is positive and decreasing. Finally

$$|S_\nu(z)| \le 2\widetilde{M} \cdot \|f\|_E \cdot \widetilde{u}_\nu(z) + \sum_{n=1}^{\nu-1} 2\widetilde{M} \cdot \|f\|_E \cdot \big(\widetilde{u}_n(z) - \widetilde{u}_{n+1}(z)\big) = 2\widetilde{M} \cdot \|f\|_E \cdot \widetilde{u}_1(z) \le 2\widetilde{M} \cdot \|f\|_E$$

and consequently

$$|\widetilde{f}(z)| = \lim_{\nu \to \infty} |S_\nu(z)| \le 2\widetilde{M} \cdot \|f\|_E,$$

$$\|\widetilde{f}\|_{\mathbb{C},0} = \|\widetilde{f}\|_{\mathbb{C}} \le 2\widetilde{M} \cdot \|f\|_E \le c_2 \cdot ||f||_E,$$

which finishes the proof of $EXP(k+1, 4)$.

Conversely let's assume that the set E admits $\mathrm{EXP}(k, u)$, i.e. it is $\mathcal{A}^\infty$-determining and

$$\begin{aligned}
&\forall f \in s(E) \quad \exists \widetilde{f} \in \mathcal{A}^\infty(E) \quad : \\
&(a) \quad \widetilde{f}_{|E} \equiv f, \\
&(b) \quad \|\widetilde{f}\|_{\mathbb{C},0} \le c_2 \cdot ||f||_E, \\
&(c) \quad \|\widetilde{f}\|_{\mathbb{C},\ell} \le (c_2 \cdot \ell^u)^{\ell + c_1} \cdot |f|_{k \cdot \ell + c_0} \quad \text{for all } \ell \in \mathbb{N},
\end{aligned}$$

with some $c_0, c_1 \ge 0$ and $c_2 \ge 1$ dependent only on the set E.

Fix arbitrarily an integer $\ell \in \mathbb{N}$, a polynomial $p \in \mathcal{P}_n$ with some $n \in \mathbb{N}$ and a point $z_0 \in E$. We can apply $\mathrm{EXP}(k, u)$ to the restriction to the set E of the polynomial $f(z) := \big(p(z) - p(z_0)\big)^\ell$, $f_{|E} \in s(E)$, to obtain the following estimate

$$\ell! \cdot |p'(z_0)|^\ell = |f^{(\ell)}(z_0)| \le \|f^{(\ell)}\|_E \le \|f\|_{E,\ell} = \|\widetilde{f}\|_{E,\ell} \le \|\widetilde{f}\|_{\mathbb{C},\ell} \le (c_2 \cdot \ell^u)^{\ell + c_1} \cdot |f_{|E}|_{k \cdot \ell + c_0},$$

because the set E is $\mathcal{A}^\infty$-determining. Now obviously we have $f \in \mathcal{P}_{\ell \cdot n}$ and therefore $\operatorname{dist}_E(f, \mathcal{P}_j) = 0$ for all $j \ge \ell \cdot n$, which implies that

$$\begin{aligned}
|f_{|E}|_{k \cdot \ell + c_0} &= \|f\|_E + \sup_{j \in \mathbb{N}} j^{k \cdot \ell + c_0} \cdot \operatorname{dist}_E(f, \mathcal{P}_j) = \\
&= \|f\|_E + \max_{j=1,\dots,\ell \cdot n - 1} j^{k \cdot \ell + c_0} \cdot \operatorname{dist}_E(f, \mathcal{P}_j) \le \|f\|_E + (\ell \cdot n - 1)^{k \cdot \ell + c_0} \cdot \max_{j=1,\dots,\ell \cdot n - 1} \operatorname{dist}_E(f, \mathcal{P}_j) \le \\
&\le \|f\|_E + (\ell \cdot n - 1)^{k \cdot \ell + c_0} \cdot \|f\|_E \le (\ell \cdot n)^{k \cdot \ell + c_0} \cdot \|f\|_E \le (\ell \cdot n)^{k \cdot \ell + c_0} \cdot \big(2 \cdot \|p\|_E\big)^\ell.
\end{aligned}$$

We combine these two estimates to obtain

$$\ell! \cdot |p'(z_0)|^\ell \le (c_2 \cdot \ell^u)^{\ell + c_1} \cdot (\ell \cdot n)^{k \cdot \ell + c_0} \cdot \big(2 \cdot \|p\|_E\big)^\ell,$$

$$|p'(z_0)| \le (c_2 \cdot \ell^u)^{1 + c_1/\ell} \cdot (\ell \cdot n)^{k + c_0/\ell} \cdot 2 \cdot \|p\|_E.$$

Because the point z_0 was arbitrary, we conclude that

$$\|p'\|_E \le M_\ell \cdot n^{k + c_0/\ell} \cdot \|p\|_E,$$

where $M_\ell := (c_2 \cdot \ell^u)^{1 + c_1/\ell} \cdot \ell^{k + c_0/\ell} \cdot 2$ depends only on the set E. This implies that the set E admits $\mathrm{GMI}(k + c_0/\ell)$, so it suffices to take ℓ sufficiently large to obtain $\mathrm{GMI}(k')$. □

Remark 10.5. Remark 9.4 applies to the previous theorem accordingly.

The following lemma is due to L. Białas-Cież. It allows to deduce a pointwise Sobolev-type inequality from a pointwise Markov-type inequality, for any function that is $\bar{\partial}$-flat in just one point.

LEMMA 10.6. *Assume that the compact set $K \subset \mathbb{C}$ admits the following inequality in the point $z_0 \in K$*

$$(1)\quad \forall 0 < r \leq 1 \quad \forall n \in \mathbb{N} \qquad \forall p \in \mathcal{P}_n \quad \forall j = 1, \ldots, n \quad : \quad |p^{(j)}(z_0)| \leq \left(\frac{c' \cdot n^{k'}}{r^{m'}} \right)^j \cdot \|p\|_{K \cap B(z_0, r)},$$

where $c', k', m' \geq 1$ depend only on the set K and possibly the point z_0. Then for any function $g \in \mathcal{C}^\infty(\mathbb{C})$ that is $\bar{\partial}$-flat in the point z_0 and $g_{|K} \not\equiv 0$ we have

$$\forall \ell \in \mathbb{N} \quad \forall j = 1, \ldots, \ell \quad : \quad \left| \frac{\partial^j g}{\partial z^j}(z_0) \right| \leq \left(2c' \cdot \ell^{k'} \right)^j \cdot ||g||_K^{1 - \frac{m' \cdot j}{\ell}} \cdot ||g||_{\mathrm{conv}\, K, \ell}^{\frac{m' \cdot j}{\ell}}.$$

PROOF. Fix $\ell \in \mathbb{N}$ and a function g as above, i.e. $D^\alpha g(z_0) = 0$ for all $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}_+^2$ such that $\alpha_2 \geq 1$. Therefore the Taylor polynomial has the following form:

$$T_{z_0}^\ell g(z) = \sum_{|\alpha| \leq \ell - 1} \frac{1}{\alpha!} \cdot D^\alpha g(z_0) \cdot (z - z_0)^\alpha = \sum_{\alpha_1 = 0}^{\ell - 1} \frac{1}{\alpha_1!} \cdot \frac{\partial^{\alpha_1} g}{\partial z^{\alpha_1}}(z_0) \cdot (z - z_0)^{\alpha_1},$$

and thus $T_{z_0}^\ell g \in \mathcal{P}_{\ell - 1}$ and $\frac{\partial^j g}{\partial z^j}(z_0) = \frac{d^j}{dz^j} T_{z_0}^\ell g(z_0)$ for each $j = 1, \ldots, \ell - 1$. We apply inequality (1) to the holomorphic polynomial $T_{z_0}^\ell g$ to obtain

$$\left| \frac{\partial^j g}{\partial z^j}(z_0) \right| = \left| \frac{d^j}{dz^j} T_{z_0}^\ell g(z_0) \right| \leq \left(\frac{c' \cdot (\ell - 1)^{k'}}{r^{m'}} \right)^j \cdot ||T_{z_0}^\ell g||_{K \cap B(z_0, r)}$$

for all $0 < r \leq 1$. Denote by $R_{z_0}^\ell g$ the Taylor remainder of the function g, i.e. $R_{z_0}^\ell g := g - T_{z_0}^\ell g$. By proposition 5.2, the Taylor formula with the remainder of Lagrange, we have for any $z_1 \in \mathbb{C}$

$$|R_{z_0}^\ell g(z_1)| \leq \min \left\{ 1, \frac{2^\ell}{\ell!} \right\} \cdot |z_1 - z_0|^\ell \cdot |g|_{[z_0, z_1], \ell}.$$

Consequently for any $0 < r \leq 1$ we have

$$\left| \frac{\partial^j g}{\partial z^j}(z_0) \right| \leq \left(\frac{c' \cdot \ell^{k'}}{r^{m'}} \right)^j \cdot \left(||g||_{K \cap B(z_0, r)} + ||R_{z_0}^\ell g||_{K \cap B(z_0, r)} \right) \leq \left(\frac{c' \cdot \ell^{k'}}{r^{m'}} \right)^j \cdot \left(||g||_K + r^\ell \cdot |g|_{\mathrm{conv}\, K, \ell} \right)$$

for $j = 1, \ldots, \ell - 1$, but obviously for $j = \ell$ this is also true.

We put $r := \left(\frac{||g||_K}{\|g\|_{\mathrm{conv}\, K, \ell}} \right)^{1/\ell} \leq 1$ to see that

$$\left| \frac{\partial^j g}{\partial z^j}(z_0) \right| \leq \left(\frac{c' \cdot \ell^{k'} \cdot ||g||_{\mathrm{conv}\, K, \ell}^{m'/\ell}}{||g||_K^{m'/\ell}} \right)^j \cdot \left(||g||_K + \frac{||g||_K}{||g||_{\mathrm{conv}\, K, \ell}} \cdot |g|_{\mathrm{conv}\, K, \ell} \right) =$$

$$= \left(c' \cdot \ell^{k'} \right)^j \cdot ||g||_{\mathrm{conv}\, K, \ell}^{\frac{m' \cdot j}{\ell}} \cdot ||g||_K^{1 - \frac{m' \cdot j}{\ell}} \cdot \left(1 + \frac{|g|_{\mathrm{conv}\, K, \ell}}{||g||_{\mathrm{conv}\, K, \ell}} \right).$$

Because $|g|_{\mathrm{conv}\, K, \ell} < \|g\|_{\mathrm{conv}\, K, \ell}$ we conclude that for $j = 1, \ldots, \ell$ we have

$$\left| \frac{\partial^j g}{\partial z^j}(z_0) \right| \leq \left(2c' \cdot \ell^{k'} \right)^j \cdot ||g||_K^{1 - \frac{m' \cdot j}{\ell}} \cdot ||g||_{\mathrm{conv}\, K, \ell}^{\frac{m' \cdot j}{\ell}}. \quad \square$$

THEOREM 10.7 [cf. Eggink, theorem 9.5; cf. Bos-Milman, theorem B]. *For any polynomially convex compact set* $E \subset\subset \mathbb{C}$ *and* $k, u, s, v \geq 1$ *we have*

$$\mathrm{EXP}(k,u) \wedge \mathrm{JP}(s,v) \implies \mathrm{SPH}(1, k\cdot s, k\cdot v + u + 1),$$
$$\mathrm{EXP}(k,u) \wedge \mathrm{WJP}(s) \implies \mathrm{WSPH}(1, k\cdot s).$$

PROOF. Let's first assume that the set E admits $\mathrm{EXP}(k,u)$ and $\mathrm{WJP}(s)$, i.e. it is $\mathcal{A}^\infty$-determining and

$$\begin{aligned}
&\forall f \in s(E) \quad \exists \widetilde{f} \in \mathcal{A}^\infty(E) \quad : \\
&(a) \quad \widetilde{f}_{|E} \equiv f, \\
&(b) \quad \|\widetilde{f}\|_{\mathbb{C},0} \leq c_2 \cdot ||f||_E, \\
&(c) \quad \|\widetilde{f}\|_{\mathbb{C},\ell} \leq (c_2 \cdot \ell^u)^{\ell + c_1} \cdot |f|_{k\cdot\ell + c_0} \quad \text{for all } \ell \in \mathbb{N},
\end{aligned}$$

with some $c_0, c_1 \geq 0$ and $c_2 \geq 1$ dependent only on the set E and additionally

$$\exists \widetilde{c}_0 \geq 0 \quad \forall \ell \geq 1 \quad \exists \widetilde{c}_\ell \geq 1 \quad \forall 0 < \delta \leq 1 \quad \forall f \in \mathcal{H}^\infty(E_\delta) \quad : \quad |f_{|E}|_\ell \leq \left(\frac{\widetilde{c}_\ell}{\delta^s}\right)^{\ell + \widetilde{c}_0} \cdot ||f||_{E_\delta}.$$

Without loss of generality we can assume that the sequence $\{\widetilde{c}_\ell\}_{\ell\in\mathbb{N}}$ is increasing.

Fix arbitrarily $0 < \delta \leq 1$, $f \in \mathcal{H}^\infty(E_\delta)$, $\ell \in \mathbb{N}$, $j \in \{1, \ldots, \ell\}$ and a point $z_0 \in E$. We can assume that $f_{|E} \not\equiv 0$, since otherwise the assertion would be trivial, because the set E is $\mathcal{A}^\infty$-determining.

By corollary 8.20 we know that $f_{|E} \in \mathcal{H}^\infty(E)_{|E} \subset s(E)$ and thus we can apply $\mathrm{EXP}(k,u)$ to find a function $\widetilde{f} \in \mathcal{A}^\infty(E)$ as above and we can combine estimate (c) with $\mathrm{WJP}(s)$ to obtain

$$\|\widetilde{f}\|_{\mathbb{C},\ell} \leq (c_2 \cdot \ell^u)^{\ell + c_1} \cdot |f_{|E}|_{k\cdot\ell + c_0} \leq (c_2 \cdot \ell^u)^{\ell + c_1} \cdot \left(\frac{\widetilde{c}_{k\cdot\ell + c_0}}{\delta^s}\right)^{k\cdot\ell + c_0 + \widetilde{c}_0} \cdot ||f||_{E_\delta}.$$

Next we apply lemma 10.6 to the set $K := E_1$ and the function $\widetilde{f}$. Because for each $0 < r \leq 1$ we have $K \cap B(z_0, r) = B(z_0, r)$, we see that the set K admits inequality (1) assumed in the lemma in each point $z_0 \in E \subset K$ with constant coefficients $c' = k' = m' = 1$. Therefore by the lemma we obtain

$$\begin{aligned}
\left|\frac{\partial^j \widetilde{f}}{\partial z^j}(z_0)\right| &\leq (2\ell)^j \cdot ||\widetilde{f}||_K^{1-\frac{j}{\ell}} \cdot ||\widetilde{f}||_{\operatorname{conv} K,\ell}^{\frac{j}{\ell}} \leq (2\ell)^j \cdot ||\widetilde{f}||_{\mathbb{C}}^{1-\frac{j}{\ell}} \cdot ||\widetilde{f}||_{\mathbb{C},\ell}^{\frac{j}{\ell}} \leq \\
&\leq (2\ell)^j \cdot c_2^{1-\frac{j}{\ell}} \cdot ||f||_E^{1-\frac{j}{\ell}} \cdot (c_2 \cdot \ell^u)^{j + c_1 \cdot j/\ell} \cdot \left(\frac{\widetilde{c}_{k\cdot\ell + c_0}}{\delta^s}\right)^{k\cdot j + (c_0 + \widetilde{c}_0)\cdot j/\ell} \cdot ||f||_{E_\delta}^{\frac{j}{\ell}} \leq \\
&\leq \left(2c_2^2 \cdot \ell^{u+1}\right)^{j + c_1} \cdot \left(\frac{\widetilde{c}_{k\cdot\ell + c_0}}{\delta^s}\right)^{k\cdot j + c_0 + \widetilde{c}_0} \cdot ||f||_E^{1-\frac{j}{\ell}} \cdot ||f||_{E_\delta}^{\frac{j}{\ell}} \leq \left(\frac{d_\ell}{\delta^{k\cdot s}}\right)^{j + d_0} \cdot ||f||_E^{1-\frac{j}{\ell}} \cdot ||f||_{E_\delta}^{\frac{j}{\ell}},
\end{aligned}$$

where $d_\ell := 2c_2^2 \cdot \ell^{u+1} \cdot \widetilde{c}_{k\cdot\ell + c_0}^{\,k}$ and $d_0 := \max\left\{c_1, \frac{c_0 + \widetilde{c}_0}{k}\right\}$ depend only on the set E.

Since E is $\mathcal{A}^\infty$-determining and $\widetilde{f}_{|E} \equiv f_{|E}$ we have $\frac{\partial^j \widetilde{f}}{\partial z^j} \equiv \frac{\partial^j f}{\partial z^j}$ on the set E and hence

$$|f|_{E,j} = |\widetilde{f}|_{E,j} = \left\|\frac{\partial^j \widetilde{f}}{\partial z^j}\right\|_E \leq \left(\frac{d_\ell}{\delta^{k\cdot s}}\right)^{j + d_0} \cdot ||f||_E^{1-\frac{j}{\ell}} \cdot ||f||_{E_\delta}^{\frac{j}{\ell}},$$

because the point $z_0 \in E$ was arbitrary. This finishes the proof of $\mathrm{WSPH}(1, k\cdot s)$.

Finally if we assume additionally that $\widetilde{c}_\ell \leq \widetilde{c}_1 \cdot \ell^v$, i.e. the set E admits $\mathrm{JP}(s,v)$, then we have

$$d_\ell \leq 2c_2^2 \cdot \ell^{u+1} \cdot \widetilde{c}_1^{\,k} \cdot (k\cdot\ell + c_0)^{k\cdot v} \leq 2c_2^2 \cdot \widetilde{c}_1^{\,k} \cdot (k + c_0)^{k\cdot v} \cdot \ell^{k\cdot v + u + 1},$$

which proves $\mathrm{SPH}(1, k\cdot s, k\cdot v + u + 1)$. $\square$

We are now ready to state the second part of our main result by simply combining theorems 10.4, 10.7 and 7.9.

THEOREM 10.8. *For any polynomially convex compact set $E \subset\subset \mathbb{C}$ and $k, s, v \geq 1$, we have the following strings of implications:*

$$\mathrm{GMI}(k) \wedge \mathrm{JP}(s,v) \implies \mathrm{EXP}(k+1,4) \wedge \mathrm{JP}(s,v) \implies \mathrm{SPH}\big(1,(k+1)\cdot s,(k+1)\cdot v+5\big) \implies \mathrm{LMP}(m',k'),$$

$$\mathrm{GMI}(k) \wedge \mathrm{WJP}(s) \implies \mathrm{EXP}(k+1,4) \wedge \mathrm{WJP}(s) \implies \mathrm{WSPH}\big(1,(k+1)\cdot s\big) \implies \mathrm{WLMP}(m'),$$

for any $m' > (k+1)\cdot s$ and $k' > (k+1)\cdot(3s+v)+5$.

Combining this additionally with theorem 8.24, corollary 8.28 and corollary 8.8, we obtain some special cases.

COROLLARY 10.9. *If a polynomially convex compact set $E \subset\subset \mathbb{C}$ admits* HCP(k) *and* ŁS(s), *with $k, s \geq 1$, then it admits* LMP(m',k') *for any $m' > (k+1)\cdot s$ and $k' > (k+1)\cdot(3s+1)+5$.*

COROLLARY 10.10. *If a compact set $E \subset\subset \mathbb{R}$ admits* HCP(k), *with $k \geq 1$, then it admits* LMP(m',k') *for any $m' > k+1$ and $k' > 4k+9$.*

COROLLARY 10.11. *If a compact set $E \subset\subset \mathbb{R}$ admits* GMI(k), *with $k \geq 1$, then it admits* LMP(m',k') *for any $m' > k+1$ and $k' > 9k+14$.*

REMARK 10.12. Compare the last corollary with [Bos-Milman, theorem B], which asserted the following string of implications for any compact set $E \subset\subset \mathbb{R}$ and $k \geq 1$:

$$\mathrm{GMI}(k) \implies \mathrm{EXP}(m,4) \implies \mathrm{SPQ}(m,k') \implies \mathrm{LMP}(m',k''),$$

for any $m' > m \geq k+4$, provided that $m \in \mathbb{N}$. No explicit statement was given concerning $k', k'' \geq 1$.

REMARK 10.13. If we know that a polynomially convex set $E \subset\subset \mathbb{C}$ admits GMI(k) and LMP(m',k') for some $k' > k \geq 1$ and $m' \geq 1$, then we could ask whether it is possible to improve the coefficient k'? Note in this context the example of the unit ball, which admits GMI(1) and LMP$(1,2)$, but does not admit LMP$(1,1)$.

REMARK 10.14. L. Białas-Cież has recently constructed a family of examples of polynomially convex compact sets in the complex plane, which admit GMI, but are not m-perfect for any $m \geq 1$ and therefore they do not admit LMP. Moreover, these sets do not admit ŁS. These examples show that without the assumption JP theorems 10.7 and 10.8 would not be true.

During our search for the weakest possible assumption, which in conjunction with GMI would allow to assert LMP for any polynomially convex compact set in the complex plane, J. Siciak presented us with the following example showing that, differently from Jackson's theorem in the complex plane 8.24, the property ŁS, and thus also JP, is not a prerequisite for Markov's properties.

EXAMPLE 10.15. Put $E := B(-2,2) \cup B(2,2)$, which is a compact set consisting of two adjacent balls. Obviously the set E is simply connected and therefore by remark 1.20 and corollary 7.7 it admits GMI(2) and LMP$(1,3)$. Consequently by theorem 10.4 and corollary 7.11 we know that it also admits EXP$(3,4)$ and SPH$(1,1,8)$. However we will show that the set E does not admit ŁS, which by proposition 8.26 implies that it does not admit JP or even WJP. Hence in theorem 10.7 the property JP, respectively WJP, is not the weakest possible assumption necessary to assert SPH, respectively WSPH.

Indeed it can be easily verified that $\psi(z) := \frac{1}{z}$ is a conformal mapping of the set $\hat{\mathbb{C}} \setminus E$ onto the belt $K := \left\{z \in \mathbb{C} \;:\; -\frac{1}{4} < \Re z < \frac{1}{4}\right\}$. Furthermore it is known that $\Psi(w) := \cot(\pi \cdot w)$ is a conformal mapping of the belt K onto the set $\hat{\mathbb{C}} \setminus B(0,1)$. Consequently by theorem 1.11.c we have for all $z \in \mathbb{C} \setminus E$

$$\Phi_E(z) = |\Psi \circ \psi(z)| = \left|\cot\frac{\pi}{z}\right| = \left|\frac{\cos \pi/z}{\sin \pi/z}\right| = \left|\frac{e^{\pi i/z} + e^{-\pi i/z}}{e^{\pi i/z} - e^{-\pi i/z}}\right| = \left|\frac{e^{2\pi i/z}+1}{e^{2\pi i/z}-1}\right|,$$

and specifically we see that for $0 < y \leq 2$ we have

$$\Phi_E(y\cdot i) = \left|\frac{e^{2\pi/y}+1}{e^{2\pi/y}-1}\right| = 1 + \frac{2}{e^{2\pi/y}-1} = 1 + \frac{2e^{-2\pi/y}}{1-e^{-2\pi/y}} < 1 + 3e^{-2\pi/y}.$$

On the other hand it is obvious that

$$\mathrm{dist}(y\cdot i, E) = \sqrt{4+y^2} - 2 = \frac{y^2}{\sqrt{4+y^2}+2} \geq \frac{y^2}{5}$$

and therefore the set E cannot admit ŁS.

We finish with an interesting application of the extension property.

PROPOSITION 10.16. *Assume that the compact set $E \subset\subset \mathbb{C}$ is the sum of two polynomially convex, disjoint compact sets, i.e. $E = A \cup B$, $A = \hat{A}$, $B = \hat{B}$ and $A \cap B = \emptyset$. Assume also that $k, u \geq 1$. If both sets A and B admit* EXP(k, u)*, then the set E admits* EXP$(k, u + 4)$.

Conversely if both sets A and B are additionally non-polar and the set E admits EXP(k, u)*, then both sets A and B admit* EXP$(k, u + k)$.

PROOF. In order to prove the first assertion, we first note that if $f \in s(E)$ then $f_{|A} \in s(A)$ and $f_{|B} \in s(B)$. Subsequently we find extensions for $f_{|A}$ and $f_{|B}$ by applying the property EXP(k, u) for the sets A and B, respectively. It is easy to see that, by using an appropriate cutoff function, we can glue these two separate extensions into one extension for the function f, meeting all the requirements of property EXP$(k, u + 4)$ for the set E.

The second assertion follows straight from lemma 8.29. Indeed we can extend an arbitrary function $f \in s(A)$ to the set B by putting $f(z) := 0$ for all $z \in B$. By the lemma we have for this extension $f \in s(E)$ and

$$\forall \ell \geq 1 \quad : \quad |f|_\ell \leq (c \cdot \ell)^\ell \cdot \left(|f_{|A}|_\ell + |f_{|B}|_\ell\right) = (c \cdot \ell)^\ell \cdot |f_{|A}|_\ell,$$

where the constant $c \geq 1$ depends only on the sets A and B. Hence we can apply property EXP(k, u) for the set E to obtain another extension $\widetilde{f} \in \mathcal{A}^\infty(E) \subset \mathcal{A}^\infty(A)$ and we see that this extension meets the requirements to assert EXP$(k, u + k)$ for the set A. Obviously an identical argument applies to the set B. $\square$

COROLLARY 10.17. *Assume that the compact set $E \subset\subset \mathbb{C}$ is the sum of two polynomially convex, disjoint, non-polar compact sets, i.e. $E = A \cup B$, $A = \hat{A}$, $B = \hat{B}$, $A \cap B = \emptyset$,* $\operatorname{cap} A > 0$ *and* $\operatorname{cap} B > 0$. *If the set E admits* GMI(k)*, where $k \geq 1$, then both sets A and B admit* GMI(k') *for any $k' > k + 1$.*

PROOF. Theorem 10.4 implies that the set E admits EXP$(k + 1, 4)$ and from proposition 10.16 it follows that the sets A and B admit EXP$(k + 1, k + 5)$. Hence we can apply theorem 10.4 again to deduce GMI(k'). $\square$

CHAPTER XI

OPEN PROBLEMS

Below is a review of some open problems related to the topics discussed in this dissertation. Some of them have been studied intensively already by other specialists in the field, while the rest of them is strictly related to new ideas introduced here.

PROBLEM 11.1. Suppose that a compact set $E \subset\subset \mathbb{C}$ is the sum of two disjoint, polynomially convex sets: $E = A \cup B$, $A = \hat{A}$, $B = \hat{B}$, $A \cap B = \emptyset$. If the set E admits GMI, does this imply that the sets A and B also admit GMI?

Corollary 10.17 solves this problem only under the additional assumption that the sets A and B are non-polar. Clearly also if the set E admits LMP, then by remark 3.2 both A and B admit LMP and thus GMI. But what happens if the set E does not admit LMP nor JP and one or two of the sets A and B is polar?

By the way, the conjecture is clearly not true if we allow the sets A and B to share even one common point.

PROBLEM 11.2. Although the class of (m, s, κ)-perfect sets defined in chapter 4 gives significant insight into the geometry of sets admitting WLMP, simultaneously we see that there remain obvious gaps to be filled. The geometric conditions contemplated in proposition 4.7 are not equivalent to (m, s, κ)-perfectness and that's why we obtain suboptimal and actually slightly awkward assertions in corollaries 4.8, 4.16 and 4.17.

So how can we characterize (m, s, κ)-perfect sets, specifically when $1 \leq s < m$? Could it be possible that sets admitting LMP(m, k) are $(m, 1, \kappa)$-perfect for all $\kappa \in \mathbb{N} \setminus \{1\}$? Interestingly, L. Białas-Cież was able to prove (personal communication, see also [Frerick, corollary 4.10], [Altun-Goncharov], [Goncharov 1], [Eggink, theorem 10.1] and [Tidten 2, theorem 2]), that $(m, 1, \kappa)$-perfect sets admit Whitney's extension property. Maybe this framework can be used to prove HCP?

In this context the work of L. Carleson and V. Totik should be mentioned. They have formulated a criterion for HCP in terms of capacities, very similar to Wiener's criterion for L-regularity. Even more interestingly, whereas for compact sets on the real axis this criterion is equivalent to HCP [Carleson-Totik, theorem 1.1], in general for sets on the complex plane an additional cone condition or quantitative condition is needed [Carleson-Totik, theorem 1.2]. These latter two conditions are clearly linked to the examples mentioned in chapter 10. See also [Siciak 4].

PROBLEM 11.3. By corollaries 2.10, 4.12 and 10.10 a compact set on the real axis that is uniformly perfect admits WSMI(1), HCP and LMP, respectively. But does it also admit SMI$(1, k)$ and LMP$(1, k)$ for some $k \geq 1$?

PROBLEM 11.4. Is it possible to construct in proposition 6.8 cutoff functions $u \in \mathcal{C}^\infty(\mathbb{C})$, which decline with the radius ϵ? If yes, then there would be no need to modify their construction in proposition 10.3 and consequently in theorem 10.4 we would have GMI$(k) \Longrightarrow$ EXP$(k, 4)$.

PROBLEM 11.5. Is it otherwise possible to improve the coefficients in theorems 7.10 and 10.8?

PROBLEM 11.6. Does each polar set in the complex plane admit JP? If yes, then this would render another proof of the fact, proven by L. Białas-Cież [Białas 2], that compact sets in the complex plane admitting GMI are not polar.

PROBLEM 11.7 [Pleśniak 1, open problems]. We still don't know whether all compact sets in the complex plane admitting GMI are L-regular. For sets on the real axis this problem was solved by the combination of the results of [Bos-Milman] and [Białas-Eggink 1]. This problem seems to be closely connected with problems 11.1 and 11.6.

Problem 11.8. Is it possible to weaken the assumptions of lemma 8.22 (and consequently theorem 8.24) by replacing HCP with GMI, without assuming L-regularity? If yes, then we would have

$$\mathrm{GMI} \wedge \text{ŁS} \implies \mathrm{GMI} \wedge \mathrm{JP} \implies \mathrm{EXP} \wedge \mathrm{JP} \implies \mathrm{SPH} \wedge \mathrm{JP} \implies \mathrm{LMP} \wedge \text{ŁS} \implies \mathrm{GMI} \wedge \text{ŁS}\,.$$

If not then there must exist a set, which admits GMI but not HCP, showing that there is no equivalence between these two properties. In order to prove this, it would be sufficient to construct a set admitting GMI and ŁS, but not admitting LMP.

Problem 11.9. Example 10.15 shows that the property JP is not the weakest possible assumption, which in conjunction with property EXP allows to assert SPH in theorem 10.7 for any polynomially convex compact set in the complex plane. In which direction should we search for such a weakest possible assumption?

A simple generalization of the property JP clearly does not do the trick. Indeed it is already defined in terms of a very narrow class of functions, i.e. functions that are holomorphic in some large open neighbourhood of a fixed compact set, while the regular norm on that neighbourhood is the most convenient possible and the dependence on δ cannot be weakened further.

Could the property that $\mathcal{A}^\infty(E)_{|E} = s(E)$ have anything to do with this? Note that if a compact set $E \subset\subset \mathbb{C}$ admits this property as well as EXP, then we can apply EXP to any function of the class $\mathcal{A}^\infty(E)_{|E}$ in order to obtain an estimate for its quotient norms in terms of its Jackson norms. Moreover, thanks to the continuity of the map $\mathcal{C}(E) \ni f \longrightarrow \mathrm{dist}_E(f, \mathcal{P}_n) \in \mathbb{R}$, for any $n \in \mathbb{Z}_+$, the space $s(E)$ is complete in its own Jackson topology. Consequently, by Banach's open mapping theorem [Rudin 1, theorem 2.11], the Jackson and quotient topologies coincide. This in turn by itself implies GMI for any set that is $\mathcal{A}^\infty$-determining, just like in the second part ($\Longleftarrow$) of the proof of theorem 9.2 [cf. Pleśniak 1, theorem 3.3.iv-v].

NOTATIONS

symbol	**stands for**
$\hat{\mathbb{C}}$	the extended complex plane
$\mathbb{R}_+$ and $\mathbb{Z}_+$	the sets of non-negative real respectively integer numbers
k, m, s, v	the main coefficients used in the analysed properties
a, b, c, d	some irrelevant constants, usually dependent on the set E
$\operatorname{int} \ell$	the integer of a positive number ℓ
$z = x + y \cdot i$	a point in the complex plane
x or t	a point on the real axis of the complex plane
$\arg z$	the argument of a complex number
r	the radius of a ball
δ	a distance
$B(z, r)$ or simply B	a closed ball centred in z with radius r
$[z_0, z_1]$	a closed interval
(m, s, κ)-perfect sets	see definition 4.5
E or K	a compact set in $\mathbb{C}$ or $\mathbb{R}$
I	an interval in $\mathbb{C}$ or $\mathbb{R}$
$\operatorname{int} E$	the interior of the set E
$\operatorname{conv} E$	the convex hull of the set E
$\hat{E}$	the polynomial hull of the set E
E_δ	a closed neighbourhood of the set E as defined in definition 1.13
$K(E, \delta)$	a closed neighbourhood of the set E as defined in definition 8.15
∂E	the boundary of the set E
∂E_t	all points for which the distance to the set E equals t
Ω	an open domain in $\mathbb{C}$
$\bar{\Omega}$	the closure of the domain Ω
$\operatorname{diam} E$	the diameter of the set E
$\operatorname{dist}(z, E)$	the distance between the point z and the set E
$\operatorname{dist}(E, K)$	the distance between the sets E and K
$\operatorname{cap} E$	the logarithmic capacity of the set E
$\mathcal{P}(\mathbb{C})$ or simply $\mathcal{P}$	the space of polynomials with complex coefficients
$\mathcal{P}_n(\mathbb{C})$ or simply $\mathcal{P}_n$	the space of polynomials of degree n or less with complex coefficients
$\mathcal{P}_n(\mathbb{R})$	the space of polynomials of degree n or less with real coefficients
$\mathcal{C}^\infty(\mathbb{C})$	the space of functions on $\mathbb{C}$ that are infinitely differentiable (smooth)
$\mathcal{A}^\infty(E)$	the space of functions of the class $\mathcal{C}^\infty(\mathbb{C})$ that are $\bar{\partial}$-flat on E
$\mathcal{H}^\infty(E)$	the space of functions of the class $\mathcal{C}^\infty(\mathbb{C})$ that are holomorphic in some open neighbourhood of the set E
$\mathcal{C}(E)$	the space of functions that are continuous on a compact set E
$s(E)$	the space of functions that can be rapidly approximated by polynomials
$\mathcal{E}(E)$	the space of Whitney fields

symbol	**stands for**
p or q	(holomorphic) polynomials
$\deg p$	the degree of polynomial p
n	the degree of a polynomial
$\|p\|_B^m$ and $\|p\|_B$	see definition 2.3
Φ_E	Siciak's extremal function of the set E
Φ_n	functions used in the definition of the extremal function
g_E	Green's function of the set $\mathbb{C} \setminus \hat{E}$ with its pole at infinity
$C(E,\rho)$	the level sets of the extremal function Φ_E
f	a function of class $\mathcal{C}^\infty(\mathbb{C})$ or narrower
$f_{\|E}$	the function f confined to the set E
$\frac{\partial f}{\partial z}$ and $\frac{\partial f}{\partial \bar{z}}$	the partial derivatives of a complex function f
$f', f^{(\ell)}$	the first and subsequent derivatives of a holomorphic function
$L_n f(\,\cdot\,)$	a Lagrange interpolation polynomial of degree n for the function f
$T_{z_0}^\ell f$	the Taylor polynomial of the function f of degree $\ell - 1$ around the point z_0
$R_{z_0}^\ell f$	its remainder
$\mathrm{dist}_E(f, \mathcal{P}_n)$	the distance on the set E between function f and $\mathcal{P}_n$
$\mathrm{supp}\, f$	the support of the function f
$\\|f\\|_E$	the usual supremum norm
$\|f\|_{E,\ell}$ and $\|\|f\|\|_{E,\ell}$	norms defined in definition 5.1
$\|\|\|f\|\|\|_{E,\ell}$	Whitney norms defined in definition 5.5
$\blacksquare f \blacksquare_{E,\ell}$	quotient norms defined in definition 6.1
$\langle\!\langle f \rangle\!\rangle_{E,\ell}$	holomorphic quotient norms defined in definition 6.5
$\|f\|_\ell$	Jackson norms defined in definition 8.1

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